Numerical methods for fractional Black-Scholes equations and variational inequalities governing option pricing

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Numerical Methods for Fractional Black-Scholes Equations and Variational Inequalities Governing Option Pricing

Wen Chen

This thesis is presented for the degree of Doctor of Philosophy of The University of Western Australia

School of Mathematics and Statistics

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Abstract

The aim of this thesis is to develop numerical methods for pricing options whose underlying follow a Lévy process, establish convergence theories for these numerical methods and demonstrate the accuracy, effectiveness and practicality of the developed methods numerically.

We first propose a second order finite difference method (FDM) for the one-dimensional fractional Black-Scholes (FBS) equation governing the valuation of European options on a single asset. We then show that both the continuous and discretized FBS equations are uniquely solvable and establish the convergence of the numerical solution generated by the FDM to the viscosity solution of the continuous FBS equation by proving that the FDM is consistent, stable and monotone. We then propose a power penalty method for a fractional-order differential linear complementarity problem (LCP) arising in the valuation of American options on a single asset. The penalty method is shown to have an exponential convergence rate depending on the penalty parameters.

Both the discretization and penalty methods for single-asset options, along with their mathematical analysis are extended to the numerical solution of the two-dimensional FBS equation and LCP arising in the valuation of European and American options on two assets. An Alternating Direction Implicit method is also proposed for solving efficiently the algebraic systems arising from the discretization of the two-dimensions FBS equation.

To demonstrate the theoretical results, extensive numerical experiments on the methods developed in this thesis are performed. Some of the test problems are solved by two existing discretization methods from the open literature. The numerical results show that our discretization methods have the 2nd-order accuracy, while the two existing methods: the Grünwald-Letnikov and L2 method have only 1st-order accuracy. The numerical results also show that the rates of convergence of the penalty methods are exponential and that our methods give accurate and financially meaningful results.
Acknowledgements

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For my dear friend Matthias Wong, thank you for letting me know more about functional analysis and love, and lots of encouragement. For my comrade Donny Lesnama, thank you for all the discussions and meetings we had together. For my forever best friend, Hongbin Xiang, thank you for being with me whenever I need you.

iv
## Contents

Abstract iii  
Acknowledgements iv  
List of Figures vii  
List of Tables ix  
Symbols xi  

1 Introduction  
  1.1 Options and Option Prices 1  
  1.2 Black-Scholes Model 2  
  1.3 Fractional Black-Scholes Model 3  
    1.3.1 European Options 6  
    1.3.2 American Options 7  
  1.4 Two-Asset Options 7  
  1.5 Numerical Methods 8  
  1.6 Research Outline 8  

2 European Options 11  
  2.1 Fractional Black-Scholes Model 11  
  2.2 The Variational Problem and Solvability 13  
  2.3 Discretization 16  
    2.3.1 Discretization of the $\alpha$-th Derivative 16  
    2.3.2 Full Discretization of (2.6a) 20  
  2.4 Convergence 22  
  2.5 Numerical Results 30  
  2.6 Conclusion 34  

3 American Options 37  
  3.1 Fractional American Option Pricing Model 37  
  3.2 The Variational Formulation and Unique Solvability 39  
  3.3 Penalty Method and Convergence 39  
    3.3.1 Penalty Method 40  
    3.3.2 Convergence 42  
  3.4 Discretization 47
3.5 Convergence of Discretization for Linear Penalty Method .......... 48
3.6 Numerical Results ............................................. 50
    3.6.1 LCP Model .................................................. 50
    3.6.2 American Option Pricing ................................. 51
3.7 Conclusion ....................................................... 55

4 European Two-Asset Options ................................. 57
    4.1 FBS Model for Two-Asset Options ....................... 57
    4.2 Variational Formulation and Unique Solvability .......... 59
    4.3 Discretization .............................................. 63
    4.4 Convergence .................................................. 66
    4.5 Alternate Direction Implicit Method ..................... 69
    4.6 Numerical Results .......................................... 72
    4.7 Conclusion .................................................... 78

5 American Two-Asset Options ................................. 81
    5.1 American Two-Asset Option Pricing Model ............... 82
    5.2 Variational Formulation and Unique Solvability .......... 83
    5.3 Penalized Equation and Convergence ..................... 84
        5.3.1 Convergence Analysis ..................................... 85
    5.4 Discretization .............................................. 89
    5.5 Numerical Results .......................................... 90
    5.6 Conclusion .................................................... 93

6 Conclusions and Future Research .......................... 95
    6.1 Conclusions ................................................... 95
    6.2 Future Research ........................................... 96
# List of Figures

2.1 Numerical investigation on $\sum_{k=0}^{200} g_k \cos((1 - k)\xi h)$ with various $\xi h$ 
2.2 Computed Value of European Call Option 
2.3 European Call Comparison 
2.4 Computed Value of European Put Option 
2.5 European Put Comparison 
3.1 Computed Value of American Put Option and Free Boundary 
3.2 American Comparison 
3.3 American and European Option Comparison 
3.4 American and European Option Comparison 2 
4.1 Computed Prices of a Call-on Min Option 
4.2 Computed Prices of a Put-on-Min Option 
4.3 Computed Prices of a Basket Option 
4.4 $C_{bs} - C_{fbs}$ Call-on-Min Options 
4.5 $P_{bs} - P_{fbs}$ Put-on-Max Options 
4.6 $V_{bs} - V_{fbs}$ Basket Options 
5.1 Computed Prices of a American Basket Option 
5.2 Basket Options Comparison
List of Tables

2.1 Computed Rates of Convergence for Example 2.1 . . . . . . . . . . . . . . . 31
2.2 Market Parameters for Example 2-3 . . . . . . . . . . . . . . . . . . . . . 31

3.1 Computed Rates of Convergence in $dx$ and $\Delta t$. . . . . . . . . . . . 51
3.2 Convergence behaviour with increasing $\lambda$. . . . . . . . . . . . . . . 52
3.3 Convergence behaviour with increasing $k$. . . . . . . . . . . . . . . . . . . 52

4.1 Error and Convergent rate of Example 4.1 . . . . . . . . . . . . . . . . . . . 73
4.2 Errors and Computational Cost of Example 4.1 . . . . . . . . . . . . . . . 74
4.3 Errors of Example 4.2 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 75
4.4 Market Parameters for a Two-Asset Option . . . . . . . . . . . . . . . . . . 75

5.1 Market Parameters for a Two-Asset American Option . . . . . . . . . . . 91
5.2 Convergence behaviour with increasing $\lambda$. . . . . . . . . . . . . . . 91
5.3 Convergence behaviour with increasing $k$. . . . . . . . . . . . . . . . . . . 92
### Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>drift rate</td>
</tr>
<tr>
<td>$r$</td>
<td>risk-free rate</td>
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<tr>
<td>$\sigma$</td>
<td>volatility</td>
</tr>
<tr>
<td>$\nu$</td>
<td>convexity adjustment</td>
</tr>
<tr>
<td>$S, S_1, S_2$</td>
<td>prices of underlying assets</td>
</tr>
<tr>
<td>$x, y$</td>
<td>natural logarithm of price of underlying assets</td>
</tr>
<tr>
<td>$t$</td>
<td>time variables</td>
</tr>
<tr>
<td>$\alpha, \beta$</td>
<td>fractional order of derivatives</td>
</tr>
<tr>
<td>$i = \sqrt{-1}$</td>
<td>imaginary number</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>characteristic function of Lévy process</td>
</tr>
</tbody>
</table>

- $U$ option value
- $V$ option value after variable transformation
- $T$ expiry date
- $K, K_1, K_2$ striking prices
- $U^*, V^*$ payoff functions
- $\hat{U}$ Fourier transform of $U$
- $X_t$ Lévy process
- $\Psi(\xi)$ characteristic exponent of the Lévy process
- $V_x$ first partial derivative of $V$ with respect to $x$
- $V_{xx}$ second partial derivative of $V$ with respect to $x$
- $\Delta$ $\partial V/\partial S$
- $\mathcal{C}$ the system discretization matrix

$$\frac{\partial^\alpha}{\partial x^\alpha}$$
<table>
<thead>
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<th>SYMBOLS</th>
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<tbody>
<tr>
<td>$-\infty D_x^\alpha$</td>
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<tr>
<td>$x_{\text{min}} D_x^\alpha$</td>
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<td>$D_h^\alpha$</td>
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<td>$-\infty D_x^\alpha U$</td>
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<td>$-\infty D_y^\beta U$</td>
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<td>$\frac{\partial^{\alpha-1} U}{\partial x^{\alpha-1}}$</td>
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</tr>
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Chapter 1

Introduction

Pricing options has attracted much attention from both mathematicians and financial engineers in the last few decades. Since then, numerous option pricing models have been developed. In this thesis, we will investigate the numerical methods for the solution of the fractional Black-Scholes (FBS) equations and variational inequality arising from option pricing. In this Chapter, we will first introduce the basis of option pricing models and option pricing approaches in the open literature. Subsequently, we will introduce our research methods and the structure of this thesis.

1.1 Options and Option Prices

A financial derivative is a financial asset whose pay-off depends on the value of the underlying assets such as stocks, indices, interest rates or foreign exchange rates, etc. An option is a derivative security which gives the owner the right but not obligation to buy (call option) or sell (put option) the underlying asset at a fixed price (strike price) \( K \) on (European type) or before (American type) a given date (maturity) \( T \). Modeling financial market with stochastic processes started in the beginning of last century. In the 1900s, Bachelier modeled stocks as a Brownian motion with drift, though this model suffers from an apparent short-coming because stock prices can be negative. It was not until 1965 that Samuelson suggested the geometric Brownian motion model. Based on this model, Black \& Scholes in 1973 [4] demonstrated that the price of a European option on a stock, whose
price follows a geometric Brownian motion with constant drift and volatility, satisfies a second order partial differential equation, known as the Black-Scholes equation.

However, this Black-Scholes (BS) model cannot correctly capture the dynamics of the option prices because the empirical data shows that the assumption of log-normal diffusion with constant volatility is not consistent with the market prices. One phenomenon that exists in all stock markets is the volatility skew or smile. To improve the performance of the BS model, several extensions of the Black-Scholes model were developed, such as the jump-diffusion models [18, 27, 32, 46], stochastic volatility models [17, 30], etc. Models based on Lévy processes were proposed and have been taken into account since late 1980s. These models can reflect some of the most important characteristics of the dynamics of stock prices which are not captured by the Gaussian model, specifically the large jump over a small time. In this thesis, we assume the stock prices follow a Lévy process.

1.2 Black-Scholes Model

The standard Black-Scholes assumed that the price of the underlying asset \( S \) follows a geometric Gaussian process

\[
\frac{dS}{S} = \mu \Delta t + \sigma dW,
\]

where \( \mu \) is the drift rate or the expected return on the asset, \( \sigma \) is the volatility and the process \( dW \) is a standard Brownian motion. By Ito’s lemma and delta hedging principle, it can be derived that the price of an European vanilla option \( V(S,t) \) is governed by the BS equation

\[
-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} + rV = 0, \quad V \in [0,\infty) \times [0,T)
\]

where \( r \) is the risk-free rate, with the terminal and boundary conditions:

\[
V(S,T) = V^*(S),
\]

\[
V(0, t) = V^*(0)e^{-r(T-t)},
\]

\[
\lim_{S \to \infty} V(S, t) = V^*(S)e^{-r(T-t)},
\]
where \( V^* \) is the pay-off function. In particular, \( V^* = [S-K]_+ \) for a European vanilla call, and \( V^* = [K-S]_+ \) for a European vanilla put, where the function \([z]_+ = \max\{z,0\}\). The BS equation can also be transformed by the new variable \( x = \ln S \):

\[
- \frac{\partial V(x,t)}{\partial t} - \frac{1}{2} \frac{\partial^2 V(x,t)}{\partial x^2} - \left( r - \frac{1}{2} \sigma^2 \right) \frac{\partial V(x,t)}{\partial x} + rV = 0
\]

The closed-form of solution is available in the literature, [39].

The American options differ from the European options in that they can be exercised at any time before the expiry date, thus the potential value of an American option is higher than that of an European option written on the same asset with the same strike price. The price \( V \) of an American option is formulated as the solution of a complementarity problem in [29].

\[
\begin{cases}
LV \geq 0, \\
V - V^* \geq 0, \\
LV \cdot (V - V^*) = 0
\end{cases}
\]

where

\[
LV := \frac{\partial V}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS \frac{\partial V}{\partial S} + rV
\]

denotes the standard BS operator, with particular boundary and terminal conditions for a specific American type option. For an American call option, it is never optimal to exercise before the expiry date on non-dividend-paying stock, however, for an American put there exists an optimal exercising time before maturity. The owner of the American put options can exercise early in order to avoid losing the time value of the options when the underlying asset price goes down to a critical value. Thus, American option pricing is a free-boundary problem whose analytical solution may not exist.

1.3 Fractional Black-Scholes Model

Black and Scholes and Merton have assumed the asset return in continuous time as a diffusion [4]. One significant shortcoming of the BS model is that the Gaussian shocks used in the model underestimate the probability that stock prices exhibit large moves or
jumps over small time interval as illustrated by real financial data. In 1976, Cox and Ross modeled it as a pure jump process which displays finite activity, and recent research considered some pure jump processes with infinity activity eg. variance gamma model by Madan and Seneta [41]. Merton proposed a jump-diffusion model where the diffusion captures frequent small moves, while jumps capture rare large moves [46]. However, jump-diffusion models either do not fit well to the real data when the number of jumps is small, or they are not very tractable when many jumps are allowed. Moreover, the empirical studies of high-frequency data suggest that the observed processes in financial markets do not have a diffusion component [6]. The stochastic volatility model increases the accuracy of pricing by using additional observable data, however, it results in higher-dimensional PDE, which has variable coefficients [26].

The Lévy process has been applied to mathematical finance for a long time. In 1960, Mandelbrot [42] suggested to use the Lévy stable (LS) distributions to model the returns in the financial market when he observed that the change of the log prices in the financial market exhibit a long-tailed distribution. Several Lévy processes have been used to model stock returns and price options [5–8, 43, 53]. In this thesis, we assume that the underlying stock price $S_t$ of an option follows, as proposed in [8], the following geometric Lévy process

$$d(\ln S_t) = (r - v)dt + dL_t$$

(1.3)

with the solution

$$S_T = S_t e^{(r - v)(T - t)} + \int_t^T dL_u,$$

where $T$ is a future known time, $r$ is the risk-free interest rate, $v$ is a convexity adjustment so that the expected value of $S_T$ becomes $E[S_T] = e^{r(T-t)}S_t$, and $dL_t$ is the increment of a Lévy process under the equivalent martingale measure (EMM). Boyarchenko and Levendorskii [6] proposed the use of a modified Lévy-stable (Lévy-α-stable) process to model the dynamics of securities. This modification introduces a damping effect in the tails of the LS distribution, which are known as KoBoL processes. A Continuous time model which allows for both diffusions and jumps of both finite and infinity activity is developed by Carr, Geman, Madan and Yor [8] is called the CGMY process including both positive and negative jumps.
Chapter 1. Introduction

A time-dependent random variable $X_t$ is a Lévy process, if and only if it has independent and stationary increments with the following log-characteristic function in Lévy-Khintchine representation

$$\ln \mathbb{E}[e^{i\xi X_t}] := t\Psi(\xi) = mit\xi - \frac{1}{2}\sigma^2 t\xi^2 + t \int_{\mathbb{R}\setminus\{0\}} (e^{i\xi x} - 1 - i\xi h(x))W(dx),$$

where $i = \sqrt{-1}$, $m \in \mathbb{R}$ is the drift rate, $\sigma \geq 0$ is the (constant) volatility, $h(x)$ is a truncation function, $W$ is the Lévy measure satisfying

$$\int_{\mathbb{R}} \min\{1, x^2\}W(dx) < \infty,$$

and $\Psi(\xi)$ is the characteristic exponent of the Lévy process which is a combination of a drift component, a Brownian motion component and a jump component. These three components are determined by the Lévy-Khintchine triplet $(m, \sigma^2, W)$. In the form of Lévy measure $W(dx) = w(x)dx$, where $w(x)$ is the Lévy density. For an LS process, the Lévy density is given by

$$w_{LS}(x) = \begin{cases} 
Dq|x|^{-1-\alpha} & \text{for } x < 0, \\
Dp|x|^{-1-\alpha} & \text{for } x > 0,
\end{cases}$$

where $D > 0$, $p, q \in [-1, 1]$ and $p + q = 1$ satisfying $0 < \alpha \leq 2$. The characteristic exponent of the LS process is

$$\Psi_{LS}(\xi) = -\frac{\sigma^\alpha}{4\cos(\alpha\pi/2)} [(1 - s)(i\xi)^\alpha + (1 + s)(-i\xi)^\alpha] + im\xi.$$

The parameters $\alpha$ and $\sigma$ are respectively the stability index and scaling parameter. The parameter $s := p - q$ is the skewness parameter satisfying $-1 \leq s \leq 1$, and $m$ is a location parameter. When $s = 1$ (resp. $s = -1$) the random variable $X$ is maximally skewed to the left (resp. right). When $\alpha = 2$ and $s = 0$, it becomes Gaussian case. A particular characteristic of the finite moment log stable process (FMLS) process is that it only exhibits downwards jumps, while upwards movements have continuous paths. The characteristic exponent of the LS process with $s = -1$, is

$$\Psi_{FMLS}(\xi) = \frac{1}{2}\sigma^\alpha \sec\left(\frac{\alpha\pi}{2}\right) (-i\xi)^\alpha,$$  

(1.4)
where $v := \frac{1}{2} \sigma^\alpha \sec \left( \frac{\alpha \pi}{2} \right)$ is the convexity adjustment of the random walk.

It is well known that when the Brownian motion in (1.1) is substituted by a Lévy process, the BS equation (1.2) becomes a partial-integro-differential equation (PIDE) \[52\].

### 1.3.1 European Options

To construct the connection between the fractional pricing equations and these processes, the authors proposed a FBS equation for the European options after Fourier transform \[9\]. It is based on the FMLS process of which the log-stock follows (1.3), with characteristic function $\Psi(\xi)$ in (1.4).

\[
\frac{\partial \hat{U}(\xi,t)}{\partial t} = \left[ r + i \xi (r - v) - \Psi(-\xi) \right] \hat{U}(\xi,t) \quad (1.5)
\]

where $\hat{U}(\xi,t)$ is the Fourier transform of $U(x,t)$. This result can be proved by the infinitesimal generator of Lévy process. The equation (1.5) generally includes all Lévy process with finite exponential moments. The different choices of $dL_t$ and the convexity adjustment $v$ will result in different FPDEs. The author in \[9\] also derived the FPDE under CGMY and KoBoL processes, which are both useful damped Lévy processes. Because of the expected value of the stock price diverges when the distribution of the random variable $X_t$ exhibits algebraic tails, the power-law truncations are not suitable for derivative pricing, whereas for FMLS it is not an issue. Therefore, we assume the risk-neutral asset $S_t$ follows an FMLS process in this paper.

Taking the inverse Fourier transform of (1.5), the price $V$ of an option of European type satisfies the following fractional partial differential equation (FPDE):

\[
L U := -\frac{\partial U}{\partial t} - \left( r + \frac{1}{2} \sigma^\alpha \sec \left( \frac{\alpha \pi}{2} \right) \right) \frac{\partial U}{\partial x} + \frac{1}{2} \sigma^\alpha \sec \left( \frac{\alpha \pi}{2} \right) \cdot [x_{\min} D_x^\alpha U] + r U = 0 \quad (1.6)
\]

for $(x,t) \in (x_{\min}, x_{\max}) \times [0,T)$ with the boundary and terminal conditions:

\[U(x_{\min},t) = U_0(t), \quad U(x_{\max},t) = U_1(t)\]
\[U(x,T) = U^*(x),\]

satisfying the compatibility conditions $U_0(T) = U^*(x_{\min})$ and $U_1(T) = U^*(x_{\max})$, where $x = \ln S$, $x_{\min} << 0$ and $x_{\max} > 0$ are two constants representing the lower and upper
Chapter 1. Introduction

bounds for $x$. In (1.6), $r$ is the risk-free rate and $\alpha \in (1, 2)$ is the order of fractional derivative, and $U_0(t), U_1(t)$ and $U^*$ are known functions. For vanilla options, $U^*(x) = [e^x - K]_+$ for a call and $U^*(x) = [K - e^x]_+$ for a put, where $K$ denotes the strike price of the options.

1.3.2 American Options

The American option pricing can be formulated as a linear complementarity problem (LCP) with a fractional BS operator. In [9], the American option price $U$ is formulated as the solution of the following LCP.

\[
\begin{align*}
\mathcal{L}U & \geq 0, \\
U - U^* & \geq 0, \\
\mathcal{L}U \cdot (U - U^*) & = 0,
\end{align*}
\]

where $\mathcal{L}$ is the fractional Black-Scholes operator defined in (1.6). Penalty methods have been successfully used for solving constrained problems in both finite and infinite dimensions [3, 51]. Recently, penalty methods have been applied for the conventional American option pricing problem and nonlinear complementarity problems in both finite and infinite dimensions [28, 34, 35, 48, 59, 63, 65]. In this thesis, we will approximate the complementarity problem by a nonlinear FBS equation based on the idea in [59] for a second-order PDEs and establish the convergence theory for this penalty method.

1.4 Two-Asset Options

Besides these famous call and put options, many exotic options have been also defined in the market. One class of the exotic options are the multi-asset options which include the basket option, rainbow option, index option, cross-currency option whose payoffs depend on two or more underlying assets. The standard two-asset model starts from the premise that the two underlying assets $S_1$ and $S_2$ follow two geometric Brownian motion processes [2, 24]. In this thesis, we assume both of the underlying assets follow the Lévy processes, thus, we extend the one-dimensional fractional Black-Scholes model into a two-dimensional Black-Scholes model.
Since the 1950s, the Alternating Direction Implicit (ADI) methods have been widely used in multi-dimensional parabolic problems because of its high computational efficiency [21, 22, 31, 50]. Therefore, we also apply this method into solving two-dimensional fractional PDEs to improve the computational efficiency.

1.5 Numerical Methods

Fractional partial differential equations as generalizations of classical integer order partial differential equations, are increasingly used to model problems in many areas such as finance, fluid flows and mechanics. Fractional spatial derivatives are used to model anomalous diffusion or dispersion in which a particle spreads at a rate inconsistent with the classical Brownian motion. Since the closed-form solutions to FPDEs of practical significance can rarely be found, various numerical techniques have been proposed for FPDEs. Fractional derivatives can be represented in different forms such as those of Riemann-Liouville (RL) and Gr"{u}wald-Letnikov (GL) [47]. Most existing discretization methods have been developed for FPDEs in GL form (cf., for example, [10–13, 20, 33, 36, 40, 44, 45, 55, 56, 62]). Using Riemann-Liouville derivative, some authors have proposed several numerical methods and analysed the space-fractional advection-dispersion equations [37, 38, 54, 61, 64].

1.6 Research Outline

In this thesis, we introduce a higher (2nd) order discretization method for the fractional derivatives to valuate European and American options under the FBS model. We further explore a new penalty method for valuation of American option on both single and two assets. We also conduct several numerical tests to verify the effectiveness of our methods.

In Chapter 2, we first introduce the fractional Black-Scholes model governing European option pricing. For the convenience of analysis, an equivalent standard form satisfying homogeneous Dirichlet boundary conditions is derived from the original one. We then reformulate this problem to a variational inequality problem and show the existence
and uniqueness of the solution to the FPDE by proving the coerciveness and continuity of the FPDE operator $L$. After that, we propose a novel discretization method for this FBS equation and establish the convergence of the numerical solution generated by the numerical scheme to the viscosity solution of the FBS equation by proving the consistency, stability and monotonicity.

In Chapter 3, we present a penalty approach to the LCP involving an FBS operator. We first formulate the LCP as a penalized FBS equation, and we show that there exists a unique viscosity solution to the penalized FPDE. We prove that the viscosity solution to the penalized FPDE converges to that of the corresponding original FBS equation as the penalty parameters approach to infinity. We also use a damped Newton’s method to solve this nonlinear FPDE. From one model LCP example, we show that the numerical discretization scheme has second order accuracy. Numerical results from the American option example demonstrate the effectiveness and accuracy of the penalty method.

In Chapter 4, we apply the discretization method developed in Chapter 2 to the two-dimensional (2D) FBS equation governing European two-asset option pricing, whose underlying follow two independent Lévy processes. We first present the mathematical model for the two-asset European options, which is 2DFBS equation. Based on the variational formulation, we prove the unique solvability of this 2DFBS equation. We establish the convergence of the numerical scheme by showing its consistency, stability and monotonicity. We also developed an ADI method for 2DFBS equation to reduce computational cost.

In Chapter 5, An American-style two-asset option pricing problem is investigated. The model of American two-asset options is described as a differential LCP involving a 2DFBS operator. We develop the power penalty method proposed in Chapter 3 into higher dimensional American option pricing problem. The convergence of the penalty method is established. We also use one numerical example to demonstrate the accuracy of this method.
In Chapter 6, we summarize this thesis and suggest the directions for future research.
Chapter 2

European Options

In this chapter, we introduce and analyze the numerical scheme of second order accuracy for fractional Black-Scholes equation governing European option pricing.

2.1 Fractional Black-Scholes Model

As mentioned in Section 1.3.1, the price of an European option satisfies the following FBS equation:

\[ \mathcal{L}U := -\frac{\partial U}{\partial t} + a \cdot \frac{\partial U}{\partial x} - b \cdot x_{\text{min}} D_{x}^{\alpha} U + rU = 0 \quad (2.1) \]

for \((x, t) \in (x_{\text{min}}, x_{\text{max}}) \times [0, T)\) with the boundary and terminal conditions:

\[ U(x_{\text{min}}, t) = U_{0}(t), \quad U(x_{\text{max}}, t) = U_{1}(t) \quad (2.2) \]
\[ U(x, T) = U^{*}(x), \quad (2.3) \]

satisfying the compatibility conditions \(U_{0}(T) = U^{*}(x_{\text{min}})\) and \(U_{1}(T) = U^{*}(x_{\text{max}})\), where \(x = \ln S\), \(x_{\text{min}} \ll 0\) and \(x_{\text{max}} > 0\) are two constants representing the lower and upper bounds for \(x\), and the coefficients

\[ a = -r - \frac{1}{2} \sigma^{2} \sec \left( \frac{\alpha \pi}{2} \right), \quad b = -\frac{1}{2} \sigma^{2} \sec \left( \frac{\alpha \pi}{2} \right). \]
Chapter 2. European Options

In (2.1), \( r \) is the risk-free rate, \( \alpha \in (1, 2) \) is the order of fractional derivative, and \( U_0(t), U_1(t) \) and \( U^* \) are known functions. For example, for vanilla options, \( U^*(x) = [e^x - K]_+ \) for a call and \( U^*(x) = [K - e^x]_+ \) for a put, where \( K \) denotes the strike price of the option.

For a given function \( W \), the \( \alpha \)-th derivative of \( W \) is given by

\[
x_0 D_x^\alpha W(x) = \frac{W(x_0)}{\Gamma(1 - \alpha)(x - x_0)\alpha} + \frac{W'(x_0)}{\Gamma(2 - \alpha)(x - x_0)\alpha - 1} + \frac{1}{\Gamma(2 - \alpha)} \int_{x_0}^{x} \frac{W''(\xi)}{(x - \xi)^{\alpha - 1}} d\xi
\]

for \( x > x_0 \), where \( x_0 \) is a given real number and \( \Gamma(\cdot) \) denotes the Gamma function. When \( W(x_0) = 0 \) and \( W'(x_0) = 0 \), it reduces to the Caputo’s representation of the fractional partial derivative. In this thesis, we adopt this definition for all fractional derivatives.

Note that the original spatial solution domain is \((-\infty, \infty)\). In computation, we truncate this infinite domain by the lower and upper bounds \( x_{\text{min}} \) and \( x_{\text{max}} \). Thus, we assume that \( x_{\text{min}} < 0 \) and \( x_{\text{max}} > 0 \) are chosen such that \( x_{\text{min}} << 0 \) and \( e^{x_{\text{max}}} >> K \).

Also note that the use of (2.4) requires \( U(x_{\text{min}}, t) = 0 \) and \( U_x(x_{\text{min}}, t) = 0 \) in order to avoid the singularity at the left boundary \( x_{\text{min}} \). Both of these can be achieved up to a truncation error, by transforming (2.1) into an FBS equation satisfying the homogeneous Dirichlet boundary conditions.

Let \( F(x, t) \) be the function defined by

\[
F(x, t) = \frac{U_1(t) - U_0(t)}{e^{x_{\text{max}}} - e^{x_{\text{min}}}} (e^x - e^{x_{\text{min}}}) + U_0(t). \tag{2.5}
\]

Clearly, \( F(x, t) \) satisfies the boundary conditions (2.2). Also, \( F \) is an exponential function of \( x \) and thus it is invariant under the 1st and \( \alpha \)-th order differentiation operations with respect to \( x \). Taking \( \mathcal{L}F \) away from both sides of (2.1) and introducing a new variable \( V(x, t) = F(x, t) - U(x, t) \), we have

\[
\mathcal{L}V(x, t) = f(x, t), \tag{2.6a}
\]

where \( f(x, t) = \mathcal{L}F \). The boundary and terminal conditions (2.2)–(2.3) then now become

\[
V(x_{\text{min}}, t) = 0 = V(x_{\text{max}}, t), \quad t \in [0, T), \tag{2.6b}
\]

\[
V(x, T) = V^*(x) := F(x, T) - U^*(x), \quad x \in (x_{\text{min}}, x_{\text{max}}). \tag{2.6c}
\]
We now consider the representation of the $\alpha$-th derivative of $V$. Since $x = \ln S$ (omitting the subscript $t$), from the definitions of $U$, $F$ and $x = \ln S$, we have

$$
\lim_{x_{\text{min}} \to -\infty} V_x(x_{\text{min}}, t) = -\lim_{x_{\text{min}} \to -\infty} \left( \frac{(U_1(t) - U_0(t))e^{x_{\text{min}}}}{e^{x_{\text{max}}} - e^{x_{\text{min}}}} + U_S(S(x_{\text{min}}), t)e^{x_{\text{min}}} \right) = 0,
$$

(2.7)
since $U_S(S(x), t)$ is bounded on $(-\infty, x_{\text{max}})$. Thus, $V_x(x_{\text{min}}, t) \to 0$ exponentially as $x_{\text{min}} \to -\infty$.

Therefore, from (2.4), (2.6b) and (2.7), we see that the fractional derivative now becomes the following Caputo’s form:

$$
x_{\text{min}}D_\alpha^x V(x, t) = \frac{1}{\Gamma(2 - \alpha)} \int_{x_{\text{min}}}^x V_{xx}(\xi, t) \frac{(x - \xi)^{\alpha - 1}}{(x - \xi)\alpha - 1} d\xi,
$$

(2.8)
up to a truncation error when $x_{\text{min}} << 0$.

### 2.2 The Variational Problem and Solvability

In this section, we consider the unique solvability of (2.6a). First, we formulate (2.6a)–(2.6c) as a variational problem, and then we show that the variational problem has a unique solution by showing its coercivity and continuity. We start this discussion by introducing some function spaces.

For an open set $\Omega \subset \mathbb{R}$ and $1 \leq p \leq \infty$, we let $L^p(\Omega) = \{ v : (\int_\Omega |v(x)|^p dx)^{1/p} < \infty \}$ denote the space of all $p$-power integrable functions on $\Omega$ equipped with the usual $L^p$-norm $\| \cdot \|_{L^p(\Omega)}$, and $(\cdot, \cdot)$ denote the usual inner product. For any $\gamma \in (0, 1]$, we let

$$H^\gamma(\mathbb{R}) := \{ u : u$ and $-\infty D_2^\gamma u \in L^2(\mathbb{R}) \}.$$

$\| \cdot \|_\gamma$ and $\| \cdot \|_{\gamma}$ are two functionals defined respectively as

$$|u|_\gamma = \| -\infty D_2^\gamma u \|_{L^2(\Omega)}, \quad \|u\|_{\gamma} = (\|u\|_{L^2(\Omega)}^2 + \| -\infty D_2^\gamma u \|_{L^2(\Omega)}^2)^{1/2},$$

for any $u \in H^\gamma(\mathbb{R})$ it is easy to show that $| \cdot |_\gamma$ and $\| \cdot \|_\gamma$ are semi-norm and norm on $H^\gamma(\mathbb{R})$ respectively. It has been shown in [19] that $H^\gamma(\mathbb{R})$ equipped with $\| \cdot \|_\gamma$ is a Sobolev space.
Similarly to the above definition of fractional Sobolev space, we define the Sobolev space of functions having a support on the finite interval $I = (x_{\min}, x_{\max})$ as

$$H^\gamma_0(I) = \{ u : u, (x_{\min} D_x^\gamma u) \in L^2(I), u(x_{\min}) = u(x_{\max}) = 0 \},$$

where $x_{\min} D_x^\gamma u$ is defined in (2.8).

In what follows, we also use $\langle \cdot, \cdot \rangle$ to denote the duality paring between $H^\gamma_0(I)$ and its dual space $H^{-\gamma}_0(I)$. Using the notations above, we pose the following problem with the boundary and initial conditions (2.6b)–(2.6c):

**Problem 2.1.** Find $V(t) \in H^{\alpha/2}_0(I)$, such that, for all $v \in H^{\alpha/2}_0(I)$,

$$\left\langle \frac{\partial V(t)}{\partial t}, v \right\rangle + A(V(t), v) = (f(t), v)$$

almost everywhere (a.e.) in $(0, T)$ satisfying terminal condition (2.6c) a.e. in $(x_{\min}, x_{\max})$, where $A(\cdot, \cdot)$ is a bilinear form defined by

$$A(V, v) = a \left\langle \frac{\partial V}{\partial x}, v \right\rangle + b \left\langle x_{\min} D_x^{\alpha-1} V, \frac{\partial v}{\partial x} \right\rangle + r(V, v), \quad V, v \in H^{\alpha/2}_0(I). \quad (2.9)$$

Using integration by parts, it is easy to verify that Problem 2.1 is the variational problem of (2.6a)–(2.6c) (cf. [19]). It has also been shown in [19] that the bilinear form $A(\cdot, \cdot)$ is coercive and continuous, as given in the following lemma:

**Lemma 2.2.** There exist positive constants $C_1$ and $C_2$, such that for any $v, w \in H^{\alpha/2}_0(I)$,

$$A(v, v) \geq C_1 \|v\|^{2/\alpha}_{\alpha/2}, \quad (2.10a)$$

$$A(v, w) \leq C_2 \|v\|^{\alpha/2}_{\alpha/2} \|w\|_{\alpha/2}, \quad (2.10b)$$

for $t \in (0, T)$ a.e..

Using this lemma, we have the following theorem.

**Theorem 2.3.** There exists a unique solution to Problem 2.1.

This theorem is just a consequence of Lemma 2.2 and Theorem 1.33 in [25], in which the unique solvability for an abstract variational problem is established. In Theorem 1.33
Chapter 2. European Options

in [25], it states that:

for the equation which is equivalent to

\[
\left\langle -\frac{\partial V(t)}{\partial t}, v \right\rangle + A(V(t), v) = (f(t), v) \quad \forall v \in H_0^{\alpha/2}(I) \quad \text{and} \quad \text{a.e. } t \in (0, T),
\]

the following existence and uniqueness result holds. Let bilinear form \( A(\cdot, \cdot) : H_0^{\alpha/2}(I) \times H_0^{\alpha/2}(I) \rightarrow \mathbb{R} \) satisfy Lemma 2.2. Then there exists a unique solution of initial-value problem for any \( f \in L^2(0, T; H_0^{-\alpha/2}(I)) \) and \( v \in H_0^{-\alpha/2} \). The proof to Theorem 2.3 is thus omitted here.

Due to the non-smoothness of the boundary condition, Problem 2.1 does not generally have a solution in the classical sense. As a result, we consider the generalised solution which satisfy the fractional differential equation almost everywhere in \( I \times (0, T) \). The further proof of the convergence of the numerical method is based on the introduction of the notation of viscosity solution. We defined the viscosity solution of equation (2.6a) as follows.

**Definition 2.4.** A function \( V \in I \times [0, T) \) is a viscosity subsolution of equation (2.6a) if, for any \( \phi \in C^1(I \times [0, T)) \),

\[
-\frac{\partial \phi(x_0, t_0)}{\partial t} + a \cdot \frac{\partial \phi(x_0, t_0)}{\partial x} - b \cdot \min_{x \in D} D_x^\alpha \phi(x_0, t_0) + r\phi(x_0, t_0) \leq 0,
\]

at any local maximum point \((x_0, t_0) \in I \times [0, T)\) of \( V - \phi \). Similarly, \( V \) is a viscosity super-solution of equation (2.6a) if for any \( \phi \in C^1(I \times [0, T)) \),

\[
-\frac{\partial \phi(x_1, t_1)}{\partial t} + a \cdot \frac{\partial \phi(x_1, t_1)}{\partial x} - b \cdot \min_{x \in \Omega} D_x^\alpha \phi(x_1, t_1) + r\phi(x_1, t_1) \geq 0,
\]

at any local minimum point \((x_1, t_1) \in \Omega \times [0, T)\) of \( V - \phi \). Then, \( V \) is a viscosity solution of equation (2.6a) if it is simultaneously a viscosity sub- and super-solution. One of the desirable features of viscosity solutions from the above definition is that classical solutions are also viscosity solution. The uniqueness of viscosity solution implies that the viscosity solution will be the classical solution if the classical solution exists.
2.3 Discretization

In this section, we consider the discretization of the fractional partial differential equation (2.1). We start with a new scheme for the discretization of the spatial fractional differential operator and then discretize the first order spatial derivative and the time derivative using the standard finite difference schemes.

2.3.1 Discretization of the $\alpha$-th Derivative

We discretize the fractional differential operator on a uniform mesh. Let the interval $I = (x_{\text{min}}, x_{\text{max}})$ be divided into $M$ sub-intervals with mesh nodes

$$x_i = x_0 + ih, \quad i = 0, 1, ..., M,$$

(2.11)

where $h = (x_{\text{max}} - x_{\text{min}})/M$, $x_0 = x_{\text{min}}$ and $x_M = x_{\text{max}}$. For clarity, we omit the variable $t$ in this subsection. When $1 < \alpha < 2$, from (2.8) we have that, for any $i \in \{1, 2, ..., M\}$,

$$x_{\text{min}} D_x^\alpha V(x_i) = \frac{1}{\Gamma(2 - \alpha)} \int_{x_{\text{min}}}^{x_i} \frac{V_{xx}(y)}{(x_i - y)^{\alpha-1}} dy$$

$$= \frac{1}{\Gamma(2 - \alpha)} \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \frac{V_{xx}(y)}{(x_i - y)^{\alpha-1}} dy$$

$$=: \frac{1}{\Gamma(2 - \alpha)} \sum_{k=1}^{i} I_{k}^i. \quad (2.12)$$

To discretize $I_{k}^i$, we first rewrite it as

$$I_{k}^i = \int_{x_{k-1}}^{x_k} \frac{V_{xx}(y) - V_{xx}(x_k)}{(x_i - y)^{\alpha-1}} dy + \int_{x_{k-1}}^{x_k} \frac{V_{xx}(x_k)}{(x_i - y)^{\alpha-1}} dy$$

$$\approx V_{xxx}(x_k) \int_{x_{k-1}}^{x_k} \frac{y - x_k}{(x_i - y)^{\alpha-1}} dy + V_{xx}(x_k) \int_{x_{k-1}}^{x_k} \frac{1}{(x_i - y)^{\alpha-1}} dy. \quad (2.13)$$

In the above, we used a truncated Taylor expansion for $V_{xx}(y) - V_{xx}(x_k)$. The derivatives $V_{xx}(x_k)$ and $V_{xxx}(x_k)$ are then approximated respectively by the following finite differences:

$$V_{xx}(x_k) \approx \delta_x^2 V_k := \frac{V_{k-1} - 2V_k + V_{k+1}}{h^2}, \quad (2.14)$$

$$V_{xxx}(x_k) \approx \delta_x^3 V_k := \frac{-V_{k-2} + 3V_{k-1} - 3V_k + V_{k+1}}{h^3}, \quad (2.15)$$
where $V_i$ denotes an approximation to $V(x_i)$ for any feasible $i$. The two integrals on the RHS of (2.13) can be evaluated exactly, and using (2.11), we found that
\[
\int_{x_{k-1}}^{x_k} \frac{1}{(x_i - y)^{\alpha-1}} dy = h^{2-\alpha} P_{i-k} \quad \text{and} \quad \int_{x_{k-1}}^{x_k} \frac{y - x_k}{(x_i - y)^{\alpha-1}} dy = h^{3-\alpha} Q_{i-k},
\]
where
\[
P_k = \frac{(k+1)^{2-\alpha} - k^{2-\alpha}}{(2-\alpha)}, \quad (2.16)
\]
\[
Q_k := \frac{(k+1)^{3-\alpha} - k^{3-\alpha}}{(2-\alpha)(3-\alpha)} - \frac{(k+1)^{2-\alpha}}{2-\alpha}. \quad (2.17)
\]

Therefore, $I_k$ can then be approximated by
\[
I_k \approx h^{-\alpha} [P_{i-k} (V_{k-1} - 2V_k + V_{k+1}) + Q_{i-k} (-V_{k-2} + 3V_{k-1} - 3V_k + V_{k+1})]. \quad (2.18)
\]

For the coefficients $P_k$ and $Q_k$, we have the following lemma:

**Lemma 2.5.** For any $i = 1, 2, ..., M$, the sequences $\{P_k\}_{k=1}^i$ and $\{Q_k\}_{k=1}^i$ satisfy
\[
0 < P_i < P_{i-1} < \cdots < P_2 < P_1,
\]
\[
Q_i > Q_{i-1} < \cdots < Q_2 < Q_1 < 0.
\]

The proof is trivial and thus it is omitted here.

Approximating $I_k$ in (2.12) by the RHS of (2.18), we define the following approximation to the $\alpha$-derivative in Caputo sense at $x_i$:
\[
x_{\min}D_x^\alpha V(x_i) = \frac{1}{\Gamma(2-\alpha)} \sum_{k=1}^{i} I_k \approx D_R^\alpha V := \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{i+1} g_k V_{i-k+1}, \quad (2.19)
\]
where $g_k$’s are given by
\[
g_0 = Q_0 + P_0,
g_1 = Q_1 - 3Q_0 + P_1 - 2P_0,
g_2 = Q_2 - 3Q_1 + 3Q_0 + P_2 - 2P_1 + P_0,
g_k = Q_k - 3Q_{k-1} + 3Q_{k-2} - Q_{k-3} + P_k - 2P_{k-1} + P_{k-2},
\]
\[\]
for \( k = 3, 4, \ldots, i + 1 \). Clearly, \( g_k \) is a weighted sum of \( Q_k, Q_{k-1}, Q_{k-2}, Q_{k-3}, P_k, P_{k-1} \) and \( P_{k-2} \). Using (2.16) and (2.17) one can derive that

\[
g_0 = \frac{1}{(2-\alpha)(3-\alpha)}, \quad \text{for } k = 0
\]

(2.20)

\[
g_1 = \frac{2^{3-\alpha} - 4}{(2-\alpha)(3-\alpha)}, \quad \text{for } k = 1
\]

(2.21)

\[
g_2 = \frac{3^{3-\alpha} - 4 \times 2^{3-\alpha} + 6}{(2-\alpha)(3-\alpha)}, \quad \text{for } k = 2
\]

(2.22)

\[
g_k = \frac{1}{(2-\alpha)(3-\alpha)} [(k+1)^{3-\alpha} - 4k^{3-\alpha} + 6(k-1)^{3-\alpha} - 4(k-2)^{3-\alpha} + (k-3)^{3-\alpha}], \quad \text{for } k = 3, 4, \ldots, i + 1
\]

(2.23)

The following lemmas establish some properties of \( g_k \).

**Lemma 2.6.** For any \( \alpha \in (1, 2) \), the coefficients \( g_k(\alpha), \quad k = 0, 1, \ldots, i + 1 \) satisfy:

1. \( g_0 > 0, \quad g_1 < 0, \quad \text{and } g_k > 0 \) for \( k = 3, 4, 5, \ldots, i + 1 \);
2. there exists an \( \alpha^* \in (1, 2) \) such that \( g_2^\alpha < 0 \) when \( \alpha \in (1, \alpha^*) \) and \( g_2(\alpha) > 0 \) when \( \alpha \in (\alpha^*, 2) \);
3. \( \sum_{k=0}^{i+1} g_k^\alpha < 0 \).

**Proof.** (1) From (2.20) to (2.22), it is easy to verify that \( g_0 > 0 \) and \( g_1 < 0 \) for any \( \alpha \in (1, 2) \).

To simplify the coefficients, let us consider \( \bar{g}_{k+1} = g_{k+1}(2-\alpha)(3-\alpha) \) for \( k \geq 2 \). From (2.23), we have

\[
\begin{align*}
\bar{g}_{k+1} &= [(k+2)^{3-\alpha} - 4(k+1)^{3-\alpha} + 6k^{3-\alpha} - 4(k-1)^{3-\alpha} + (k-2)^{3-\alpha}] \\
&= [(k+2)^{3-\alpha} - 3(k+1)^{3-\alpha} + 3k^{3-\alpha} - (k-1)^{3-\alpha}] \\
&\quad - [(k+1)^{3-\alpha} - 3k^{3-\alpha} + 3(k-1)^{3-\alpha} - (k-2)^{3-\alpha}] \\
&=: f_1(k+1) - f_1(k).
\end{align*}
\]
To show $\bar{g}_{k+1} > 0$, it suffices to show $f_1(k)$ is strictly increasing, which is equivalent to showing that $f_1'(k) > 0$. Differentiating $f_1$ with respect to $k$ gives

$$f_1'(k+1) = (3 - \alpha) \left[ (k + 2)^{2-\alpha} - 3(k + 1)^{2-\alpha} + 3k^{2-\alpha} - (k - 1)^{2-\alpha} \right]$$

$$= (3 - \alpha) \left\{ [(k + 2)^{2-\alpha} - 2(k + 1)^{2-\alpha} + k^{2-\alpha}] - [(k + 1)^{2-\alpha} - 2k^{2-\alpha} + (k - 1)^{2-\alpha}] \right\}$$

$$= (3 - \alpha)(f_2(k+1) - f_2(k)).$$

We now show $f_2(k)$ is also strictly increasing. Differentiating $f_2(k)$, we have

$$f_2'(k) = (2 - \alpha) \left[ (k + 1)^{1-\alpha} - 2k^{1-\alpha} + (k - 1)^{1-\alpha} \right]$$

$$= (2 - \alpha) \left\{ [(k + 1)^{1-\alpha} - k^{1-\alpha}] - [k^{1-\alpha} - (k - 1)^{1-\alpha}] \right\}$$

$$= (2 - \alpha)(f_3(k) - f_3(k-1)).$$

It remains to prove $f_3$ is strictly increasing. Differentiating $f_3(k)$, we have

$$f_3'(k) = (1 - \alpha) \left[ (k + 1)^{-\alpha} - k^{-\alpha} \right] > 0,$$

where $1 < \alpha < 2$.

Therefore, $\bar{g}_{k+1} > 0$ for $k \geq 2$, or, $g_k = \bar{g}_k/[(2 - \alpha)(3 - \alpha)] > 0$ for $k \geq 3$.

(2) The proof of this is trivial so we omit it.

(3) For the finite difference scheme in (2.19), the approximation of the $\alpha$-th derivative of $V(x) = 1$ becomes exact if $x_{\text{min}} \to -\infty$, i.e.,

$$D_h^\alpha 1 = \frac{h^{-\alpha}}{\Gamma(2 - \alpha)} \sum_{k=0}^{\infty} g_k = 0.$$

Therefore we can have $\sum_{k=0}^{\infty} g_k = 0$. Since $g_k > 0$, $k \geq 3$, then we have $\sum_{k=1+2}^{\infty} g_k > 0$, so the partial sum $\sum_{k=0}^{i+1} g_k < 0$. \qed

**Remark 2.7.** We comment that $\alpha^*$ defined in Lemma 2.6 is the root of $g_2(\alpha)$ in $(1, 2)$. Using Matlab we find that $\alpha^* \approx 1.5546$. 


2.3.2 Full Discretization of (2.6a)

For a positive integer \( N \), let \((0, T)\) be divided into \( N \) sub-intervals with the mesh points

\[ t_j = (N - j)\Delta t, \quad j = 0, 1, ..., N, \]

where \( \Delta t = T/N \). Using the central differencing for the first derivative in space, Crank-Nicolson time stepping method and the scheme (2.19) for the \( \alpha \)-th derivative, we construct the following discretization scheme for (2.6a) and define the discretized FBS operator:

\[
\mathcal{L}_{h, \Delta t} V^j_i := \frac{V^j_i - V^j_{i-1}}{-\Delta t} + \frac{1}{2} \left( -a \frac{V^j_{i+1} - V^j_{i-1}}{2h} + \frac{b}{\Gamma(2-\alpha)h^\alpha} \sum_{k=0}^{i+1} g_k V^j_{i+1-k} - rV^j_i \right) + \frac{1}{2} \left( -a \frac{V^j_{i+1} - V^j_{i-1}}{2h} + \frac{b}{\Gamma(2-\alpha)h^\alpha} \sum_{k=0}^{i+1} g_k V^j_{i+1-k} - rV^j_i \right) = \frac{1}{2} (f^j_i + f^{j+1}_i),
\]

(2.24)

with the boundary and terminal conditions:

\[ V^0_j = 0 = V^j_M, \quad V^0_i = V^*(ih), \]

(2.25)

for \( i = 1, 2, ..., M - 1 \) and \( j = 0, 2, ..., N - 1 \), where \( V^j_i \) denotes an approximation to \( V(x_i, t_j) \), \( f^k_i = f(x_i, t_k) \) for \( k = j \) and \( j + 1 \), and the initial condition \( V^* \) is given in (2.6c).

Let \( \mu = -b \frac{\Delta t}{\Gamma(2-\alpha)h^\alpha} \) and \( \eta = a \frac{\Delta t}{2h} \), we rewrite equation (2.24) as

\[
\begin{align*}
&\left[ 1 + \frac{1}{2} (\mu g_1 + r \Delta t) \right] V^{j+1}_i - \frac{1}{2} \left[ (\eta + \mu g_0) V^{j+1}_{i+1} + (-\eta + \mu g_2) V^{j+1}_{i-1} + \mu \sum_{k=0}^{i+1} g_k V^{j+1}_{i+1-k} \right] \\
= &\left[ 1 - \frac{1}{2} (\mu g_1 + r \Delta t) \right] V^j_i - \frac{1}{2} \left[ (\eta + \mu g_0) V^j_{i+1} + (-\eta + \mu g_2) V^j_{i-1} + \mu \sum_{k=3}^{i+1} g_k V^j_{i+1-k} \right] \\
&+ \frac{\Delta t}{2} (f^j_i + f^{j+1}_i),
\end{align*}
\]

for \( j = 0, 1, ..., N - 1 \).

This linear system can be further written as the following matrix equation:

\[
\left( I + \frac{1}{2} C \right) \bar{V}^{j+1} = \left( I - \frac{1}{2} C \right) \bar{V}^j + \frac{\Delta t}{2} \left( \bar{f}^{j+1} + \bar{f}^j \right),
\]

(2.26)
where $\vec{V}^k = (V_1^k, V_2^k, \ldots, V_{M-1}^k)^T$ and $\vec{f}^k = (f_1^k, f_2^k, \ldots, f_{M-1}^k)^T$ are $M - 1$ column vectors, for $k = j, j+1$. The system matrix $\mathbf{C} = (c_{ij})$ an $(M - 1) \times (M - 1)$ is matrix of the form

$$\mathbf{C} = \mathbf{G} + \mathbf{B} + r\Delta t \mathbf{I},$$

(2.27)

where $\mathbf{I}$ denotes the $(M - 1)$-dimensional identity matrix,

$$G = \mu \begin{bmatrix} g_1 & g_0 & 0 & 0 & \cdots & 0 \\ g_2 & g_1 & g_0 & 0 & \cdots & 0 \\ g_3 & g_2 & g_1 & g_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ g_{M-2} & g_{M-3} & \cdots & g_2 & g_1 & g_0 \\ g_{M-1} & g_{M-2} & \cdots & g_3 & g_2 & g_1 \end{bmatrix},$$

$$B = \eta \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 0 \end{bmatrix}_{(M-1)\times(M-1)}.$$

Clearly, $G$ and $B$ arise respectively from the discretization of the $\alpha$-th derivative and the first derivative $\partial V/\partial x$. It is trivial to show that the elements $c_{ij}$ are given by

$$c_{ij} = \begin{cases} 
\mu g_0 + \eta, & j = i + 1, \\
\mu g_1 + r\Delta t, & j = i, \\
\mu g_2 - \eta, & j = i - 1, \\
\mu g_k, & j = i - k + 1, \quad k = 3, 4, \ldots, i, \\
0, & \text{otherwise.}
\end{cases}$$

We comment that $\mathbf{C}$ is a Toeplitz matrix as the elements in each diagonal of $\mathbf{C}$ are constants.
2.4 Convergence

In this section, we show that the solution to (2.24) converges to the viscosity solution to (2.1) by proving that the numerical scheme proposed in the previous section is consistent, stable and monotone. We start with the following theorem:

**Theorem 2.8. (Consistency)** The finite difference scheme for (2.1), defined by (2.24), is consistent, with a truncation error of order \( \mathcal{O}(\Delta t^2 + h^2) \).

**Proof.** In what follows, we let \( C \) denote a generic positive constant, independent of \( h \) and \( \Delta t \). We first consider the truncation error in the approximation \( D^\alpha_h \) to \( D^\alpha \) at \( x_i \) for any \( i = 1, 2, ..., M - 1 \). From (2.12) and (2.13) we have that, for any function \( V(x) \) sufficiently smooth on \( I \),

\[
x_0 D^\alpha_h V(x_i) = \frac{1}{\Gamma(2 - \alpha)} \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \frac{V''(y)}{(x_i - y)^{\alpha-1}} dy
\]

\[
= \frac{1}{\Gamma(2 - \alpha)} \sum_{k=1}^{i} \left[ \int_{x_{k-1}}^{x_k} V''(y) - V''(x_k) \frac{y - x_k}{(x_i - y)^{\alpha-1}} dy + \int_{x_{k-1}}^{x_k} V''(x_k) \frac{1}{(x_i - y)^{\alpha-1}} dy \right] + E_i,
\]

where \( E_i \) denotes the following remainder in the approximation of \( V''(y) - V''(x_{k-1}) \) by a truncated Taylor’s expansion:

\[
E_i = \frac{1}{\Gamma(2 - \alpha)} \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \frac{V^{(4)}(\xi_k)}{2} \frac{(y - x_k)^2}{(x_i - y)^{\alpha-1}} dy,
\]

for \( \xi_k \in (x_{k-1}, x_k) \) and \( i = 1, 2, ..., M - 1 \). From this equality we have the following estimation,

\[
|E_i| \leq \|V^{(4)}\|_\infty \frac{h^2}{2\Gamma(2 - \alpha)} \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \frac{1}{(x_i - y)^{1-\alpha}} dy
\]

\[
\leq \frac{h^2}{2\Gamma(2 - \alpha)} \sum_{k=1}^{i} \int_{x_{k-1}}^{x_k} \frac{1}{(x_i - y)^{1-\alpha}} dy
\]

\[
= \frac{h^2}{2\Gamma(3 - \alpha)} (x_i - x_0)^{2-\alpha}
\]

\[
\leq Ch^2
\]

(2.28)
for any $i = 1, 2, ..., M - 1$.

It is standard to verify that the finite difference operators in (2.14) and (2.15) satisfy

$$|V''(x_k) - \delta^2_x V(x_k)| \leq C h^2,$$  \hfill (2.29)

$$|V'''(x_k) - \delta^3_x V(x_k)| \leq C h.$$  \hfill (2.29)

From (2.28), (2.29) and the fact that $y - x_{k-1} < h$, the truncation error in the discretization of the $\alpha$-th derivative is bounded by

$$\left| x_0 D^\alpha_{x_i} V(x_i) - D^\alpha_{x_i} V(x_i) \right| \leq C \left( h^2 + \int_{x_0}^{x_i} \frac{1}{(x_i - y)^{\alpha-1}} dy \right) \leq C h^2,$$  \hfill (2.30)

where $C$ is a generic constant, as the last integral in the above expression exists and is bounded. Finally, it is well-known that Crank-Nicolson time stepping scheme, the central differencing and the mid-point quadrature rule used in (2.24) are all at least 2nd-order accurate on uniform meshes. Combining this fact with (2.30), we have the following estimation for the error between the FBS operator and the discretized FBS operator:

$$|\mathcal{L}V(x_i, t) - \mathcal{L}_{h, \Delta t} V(x_i, t)| \leq C (h^2 + \Delta t^2).$$

The truncation error is of second order in time and space, therefore, the discretization is consistent.

**Theorem 2.9. (Stability)** The finite difference scheme defined by (2.24) is unconditionally stable.
Chapter 2. European Options

**Proof.** We use the semi-discrete Fourier transform to prove the stability of the Crank-Nicolson method. Using \( \mu \) and \( \eta \) introduced in Subsection 2.3.2, we rewrite (2.24) as

\[
V_i^{j+1} - V_i^j + \frac{1}{2} \left[ \eta \left( V_i^{j+1} - V_i^{j-1} \right) + \mu \sum_{k=0}^\infty g_k V_{i-k}^{j+1} + r \Delta t V_i^{j+1} \right] \\
+ \frac{1}{2} \left[ \eta \left( V_i^{j+1} - V_i^{j-1} \right) + \mu \sum_{k=0}^\infty g_k V_{i-k}^{j} + r \Delta t V_i^{j} \right] = \Delta t f_i^{j+1/2},
\]

(2.31)

for \( i = 1, 2, ..., M - 1 \) and any feasible \( j \), where \( f_i^{j+1/2} = (f_i^j + f_i^{j+1})/2 \). This system has the matrix form (2.26). From the definition (2.27) we see that all the coefficient matrices in (2.26) are Toeplitz matrices. Thus, each of the terms in (2.26) can be written as convolution of some coefficient vectors with a finite support and one of \((\cdots, 0, V_1^n, \cdots, V_M^n, 0, \cdots)\) for \( n = j \) and \( n = j + 1 \) and \((\cdots, 0, f_1^{j+1/2}, \cdots, f_M^{j+1/2}, 0, \cdots)\), where \( V_i^n = 0 \) for all \( j = 1, 2, ..., N, i \leq 0 \) and \( i \geq M \). Applying the discrete Fourier transform via the semidiscrete Fourier transform pair

\[
\hat{V}_i^j = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ix} \hat{V}^j(x) dx, \\
\hat{V}^j(x) = h \sum_{i=-\infty}^{\infty} V_i^j e^{-ix},
\]

to the system , or equivalently replacing \( V_i^n \) and \( f_i^n \) in (2.32) with \( \hat{V}^j e^{m\xi h} \) and \( \hat{f}^{n+1/2} e^{m\xi h} \) respectively for all admissible \( m \) and \( n \), we have

\[
(\hat{V}^{j+1} - \hat{V}^j) e^{i\xi h} + \frac{1}{2} e^{i\xi h} \hat{V}^{j+1} \left[ \eta(e^{\xi h} - e^{-\xi h}) + \mu \sum_{k=0}^{i+1} g_k e^{\xi(-k+1)h} + r \Delta t \right] \\
+ \frac{1}{2} e^{i\xi h} \hat{V}^j \left[ \eta(e^{\xi h} - e^{-\xi h}) + \mu \sum_{k=0}^{i+1} g_k e^{\xi(-k+1)h} + r \Delta t \right] = \Delta t \hat{f}^{j+1/2},
\]

(2.32)

where \( \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right] \) and \( \hat{V}^n \) and \( \hat{f}^n \) are respectively the discrete Fourier transform of \( V^n \) and \( f^n \) for \( n = j \) and \( j + 1 \). Dividing both sides by \( e^{i\xi h} \) and rearranging the resulting
Chapter 2. European Options

equation, we get

\[
\hat{V}^{j+1} = \hat{V}^j + \frac{1}{2} \left[ \eta (e^{\xi h_i} - e^{-\xi h_i}) + \mu \sum_{k=0}^{i+1} g_k e^{\xi (-k+1)h_i} + r \Delta t \right] \\
+ \frac{\Delta t \hat{f}^{j+1/2}}{1 + \frac{1}{2} \left[ \eta (e^{\xi h_i} - e^{-\xi h_i}) + \mu \sum_{k=0}^{i+1} g_k e^{\xi (-k+1)h_i} + r \Delta t \right]} \\
= \hat{V}^j - (A + Bi) + \frac{\Delta t}{1 + (A + Bi)} \hat{f}^{j+1/2},
\]

(2.33)

where

\[
A = \frac{\mu}{2} \sum_{k=0}^{i+1} g_k \cos ((1-k)\xi h) + \frac{r \Delta t}{2} , \quad B = \frac{\eta}{2} \sin (\xi h) + \frac{\mu}{2} \sum_{k=0}^{i+1} g_k \sin ((1-k)\xi h).
\]

For stability, we are going to show that

\[
\left| \frac{1 - (A + Bi)}{1 + (A + Bi)} \right| \leq 1 + C \Delta t.
\]

(2.34)

It is trivial when \( A \geq 0 \). Since \( \mu = \frac{-b \Delta t}{\Gamma(2-\alpha) h^\alpha} < 0 \), so we are going to prove \( \sum_{k=0}^{\infty} g_k \cos((1-k)\xi h) \leq 0 \).

From (3) in Lemma 2.6, we have

\[
-g_1 \geq \sum_{k=0, k \neq 1}^{\infty} g_k,
\]

(2.35)

with \( g_k > 0 \) when \( k > 3 \). From (2.20) and (2.22), we have that \( g_0 + g_2 > 0 \). Therefore, we can derive the following estimation,

\[
\sum_{k=0}^{\infty} g_k \cos((1-k)\xi h) = g_0 \cos(\xi h) + g_1 \cos(0) + g_2 \cos(-\xi h) + g_3(-2\xi h) + \cdots + g_{i+1} \cos(-i\xi h) + \cdots \\
= g_1 + (g_0 + g_2) \cos(\xi h) + \sum_{k=3}^{\infty} g_k \cos((k-1)\xi h) \\
\leq \sum_{k=0}^{\infty} g_k \leq 0.
\]
Chapter 2. *European Options*

As a result, we can get

\[
A = \frac{\mu}{2} \sum_{k=0}^{i+1} g_k \cos ((1 - k)\xi h) + \frac{r\Delta t}{2} \geq 0. \tag{2.37}
\]

The magnitude of multiplication factor in (2.33) is less than 1 for all \(\xi \in [\pi/h, \pi/h]\), then the bound in (2.34) is trivial with \(C = 0\).

Numerical investigation on the estimate of \(\sum_{k=0}^{\infty} g_k \cos((1 - k)\xi h)\) with various \(\xi\) and \(h\) demonstrates the same results. Since \(\xi \in [-\pi/h, \pi/h]\), so we can choose different \(\xi h \in [-\pi, \pi]\) and \(i = 200\) and observe the behaviour of \(\sum_{k=0}^{i} g_k \cos((1 - k)\xi h)\) which is shown in the following figure 2.1.

This figure illustrate that the function is negative on the domain \([0, \pi]\). Since the sequence \(\{\sum_{k=0}^{i} g_k \cos((1 - k)\xi h)\}\) is a also monotonic decreasing Cauchy sequence, the limit exists when \(i \to \infty\). We can have \(\sum_{k=0}^{\infty} g_k \cos((1 - k)\xi h) < 0\). As a result, using (2.34), we have

![Figure 2.1: Numerical investigation on \(\sum_{k=0}^{200} g_k \cos((1 - k)\xi h)\) with various \(\xi h\)](image)
from (2.33) when ∆t is sufficiently small,
\[
\begin{align*}
|\hat{V}^{j+1}| &\leq |\hat{V}^{j}| + \Delta t \left| f^{j+1/2} \right|
\leq |\hat{V}^{j-1}| + \Delta t \left[ \sum_{k=0}^{j} \left| f^{k+1/2} \right| \right]
\leq \cdots \leq |\hat{V}^0| + \Delta t \sum_{k=0}^{j} \left| f^{k+1/2} \right|
\leq |\hat{V}^0| + \frac{1}{N} \sum_{k=0}^{j} \left| f^{k+1/2} \right|.
\end{align*}
\]
Using Cauchy-Schwarz inequality, we have
\[
\begin{align*}
|\hat{V}^{j+1}|^2 &\leq |\hat{V}^0|^2 + \frac{j}{N^2} \sum_{k=0}^{j} \left| f^{k+1/2} \right|^2 \leq |\hat{V}^0|^2 + \frac{1}{N} \sum_{k=0}^{j} \left| f^{k+1/2} \right|^2,
\end{align*}
\]
for any \( j \leq N - 1 \). \( \hat{V}^{j+1}, \hat{V}^0 \) and \( f^{k+1/2} \) are all functions of \( \xi \in [-\pi/h, \pi/h] \). For any function \( W \in H_0^{3/2}(I) \), let \( ||W||_{0,h} = \left( \sum_{i=M}^{M-1} |W_i|^2 \right)^{1/2} \) denote the discrete \( L^2 \)-norm of \( W \). Using the properties of the discrete Fourier and its inverse transforms (particularly Parseval’s equality) we have
\[
||V^{j+1}||_{0,h}^2 = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} |\hat{V}^{j+1}|^2 d\xi \leq \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} |\hat{V}^0|^2 d\xi + \frac{1}{N} \sum_{k=0}^{j} \int_{-\pi/h}^{\pi/h} |\hat{f}^{k+1/2}|^2 d\xi
= L \left( ||V^0||_{0,h}^2 + \frac{1}{N} \sum_{k=0}^{j} ||f^{k+1/2}||_{0,h}^2 \right)
\leq L \left( ||V^0||_{0,h}^2 + ||f||_{L^\infty(I \times (0,T))}^2 \right),
\]
from which we have (recall \( L \) is a generic positive constant)
\[
||V^{j+1}||_{0,h} \leq \bar{L} \left( ||V^0||_{0,h} + ||f||_{L^\infty(I \times (0,T))} \right).
\]
Therefore, the numerical method is unconditionally stable.

We now show that the numerical scheme is monotone.

**Theorem 2.10.** (Monotonicity) The discretization scheme established in (2.24) is monotone when \( \Delta t \leq \frac{2}{r} \).
Chapter 2. European Options

Proof. Let

$$F_{i}^{j+1} \left( V_{i}^{j+1}, V_{i+1}^{j+1}, V_{i-1}^{j+1}, \ldots, V_{0}^{j+1}, V_{0}^{j}, V_{i+1}^{j}, V_{i-1}^{j}, \ldots, V_{0}^{j} \right)$$

$$:= \left[ 1 + \frac{1}{2} (\mu g_{1} + r \Delta t) \right] V_{i}^{j+1} + \frac{1}{2} (\eta + \mu g_{0}) V_{i+1}^{j+1} - \frac{1}{2} (\eta - \mu g_{2}) V_{i-1}^{j+1} + \frac{1}{2} \mu \sum_{k=3}^{i+1} g_{k} V_{i-k+1}^{j+1}$$

$$- \left[ 1 - \frac{1}{2} (\mu g_{1} + r \Delta t) \right] V_{i}^{j} + \frac{1}{2} (\eta + \mu g_{0}) V_{i+1}^{j} - \frac{1}{2} (\eta - \mu g_{2}) V_{i-1}^{j} + \frac{1}{2} \mu \sum_{k=3}^{i+1} g_{k} V_{i-k+1}^{j}.$$  

To prove that $F_{i}^{j+1}$ is monotone, we first need to show that

$$\sum_{k=0}^{i+1} g_{k} - \frac{1}{2} g_{1} > 0.$$

For simplicity, we let $\beta = 3 - \alpha$. From the property of the coefficients (2.20)–(2.23) we have

$$\sum_{k=0}^{3} g_{k} - \frac{1}{2} g_{1} = g_{0} \left[ -1 + 3 \times 2^{\beta} - 3 \times 3^{\beta} + 4^{\beta} - \frac{1}{2} \times (2^{\beta} - 4) \right] = g_{0} f(\beta),$$

where $f(\beta) := (1 + 2.5 \times 2^{\beta} - 3 \times 3^{\beta} + 4^{\beta})$. It now suffices to show that $f(\beta) > 0$ for $\beta \in (1, 2)$. Since $f(2) = 0$, we only need to show that $f(\beta)$ is strictly decreasing for $\beta \in (1, 2)$. Differentiating $f$ gives

$$f'(\beta) = 2.5 \times \ln(2) \times 2^{\beta} - 3 \times \ln(3) \times 3^{\beta} + \ln(4) \times 4^{\beta}.$$  

It is easy to show, even graphically, that $f'(\beta) < 0$ for all $\beta \in [1, 2]$. Therefore,

$$\sum_{k=0}^{3} g_{k} - \frac{1}{2} g_{1} > 0.$$  

From Lemma 2.6, we have that $g_{k} > 0$ for $k > 3$. Thus, when $i \geq 2$,

$$\sum_{k=0}^{i+1} g_{k} - \frac{1}{2} g_{1} \geq \sum_{k=0}^{3} g_{k} - \frac{1}{2} g_{1} > 0.$$  

28
Crank-Nicolson’s time-stepping scheme, they hold true for a general two-level time-stepping scheme. The consequence of the results established in [1, 15, 16].

Partial integro-differential equations (PIDEs). Since (2.1) is a PIDE, Theorem 2.11 is applicable to the viscosity solution of the PDE. In [15] and [16], Cont and Tankov extended this result to more general cases.

Numerical schemes. Barles and Souganidis showed in [1] that any finite difference scheme approximating a general nonlinear 2nd-order PDE which is locally consistent, stable and monotone generates a solution converging uniformly on a compact subset of $[0, T] \times \mathbb{R}$ to the unique viscosity solution of the PDE. In [15] and [16], Cont and Tankov extended this result to partial integro-differential equations (PIDEs). Since (2.1) is an PIDE, Theorem 2.11 is applicable to the two-level time-stepping scheme.

We now use the above result to prove the monotonicity of $F^{j+1}_i$. When $\Delta t \leq \frac{2}{r}$, we have from the definition of $F^{j+1}_i$ that, for any $\varepsilon > 0$ and feasible $i$ and $j$,

$$F^{j+1}_i \left( V^{j+1}_i, V^{j+1}_{i+1}, \ldots, V^{j+1}_{i-1}, V^j_i, V^j_{i+1}, \ldots, V^j_{i-1}, V^0_i \right)$$

$$= F^{j+1}_i \left( V^{j+1}_i, V^{j+1}_{i+1}, \ldots, V^j_i, V^j_{i+1}, \ldots, V^j_{i-1}, V^0_i \right)$$

$$- \left[ 1 - \frac{1}{2} \left( \mu g_1 + r \Delta t \right) \right] \varepsilon + (\eta + \mu g_0) \varepsilon - (\eta - \mu g_2) \varepsilon + \mu \sum_{k=3}^{i+1} g_k \varepsilon$$

$$\leq F^{j+1}_i \left( V^{j+1}_i, V^j_{i+1}, \ldots, V^j_{i-1}, V^0_i \right) + \mu \left( \sum_{k=0}^{i+1} g_k \right) \varepsilon - \left( 1 - \frac{1}{2} r \Delta t \right) \varepsilon$$

$$\leq F^{j+1}_i \left( V^{j+1}_i, V^j_{i+1}, \ldots, V^0_i \right),$$

since $\mu < 0$.

Furthermore, since $g_1 < 0$, so $\mu g_1 > 0$, the following inequality holds.

$$F^{j+1}_i \left( V^j_i, V^{j+1}_{i+1}, \ldots, V^{j+1}_{i-1}, V^j_i, V^j_{i+1}, \ldots, V^j_{i-1}, V^0_i \right)$$

$$= F^{j+1}_i \left( V^j_i, V^{j+1}_{i+1}, \ldots, V^{j+1}_{i-1}, V^j_i, V^j_{i+1}, \ldots, V^j_{i-1}, V^0_i \right) + \left[ 1 + \frac{1}{2} \left( \mu g_1 + r \Delta t \right) \right] \varepsilon$$

$$> F^{j+1}_i \left( V^j_i, V^{j+1}_{i+1}, \ldots, V^{j+1}_{i-1}, V^j_i, V^j_{i+1}, \ldots, V^j_{i-1}, V^0_i \right).$$

The two inequalities above show that the scheme is monotone.

Combining Theorems 2.8, 2.9 and 2.10, we have the following convergence result.

**Theorem 2.11. (Convergence)** Let $V$ be the viscosity solution to (2.6a) – (2.6c) and $V_{h, \Delta t}$ be the solution to (2.24) – (2.25). Then, $V_{h, \Delta t}$ converges to $V$ as $(h, \Delta t) \to (0, 0)$.

In fact, conventionally, Theorems 2.8 and 2.9 already imply the convergence of our numerical scheme. Barles and Souganidis showed in [1] that any finite difference scheme for a general nonlinear 2nd-order PDE which is locally consistent, stable and monotone generates a solution converging uniformly on a compact subset of $[0, T] \times \mathbb{R}$ to the unique viscosity solution of the PDE. In [15] and [16], Cont and Tankov extended this result to partial integro-differential equations (PIDEs). Since (2.1) is an PIDE, Theorem 2.11 is the consequence of the results established in [1, 15, 16].

We comment that though the theoretical results in this section are established for Crank-Nicolson’s time-stepping scheme, they hold true for a general two-level time-stepping scheme.
scheme with a splitting parameter $\theta \in [0.5, 1]$. However, when $\theta \in (0.5, 1]$, the truncation error in Theorem 2.8 is of the order $O(\Delta t + h^2)$ instead of $O(\Delta t^2 + h^2)$. For clarity, we omit this discussion here.

### 2.5 Numerical Results

In this section, we first present one model example with known exact solution to demonstrate the second order rate of convergence of our numerical scheme. And meanwhile, we compare our method with another two famous discretization methods. After that we apply this discretization scheme to solve two FBS equations governing respectively European call and put options and analyze the numerical results. All the computations have been performed under MATLAB environment.

**Example 2.1.** A fractional diffusion equation with non-homogeneous boundary conditions:

$$
\frac{\partial u(x, t)}{\partial t} - 5 D_{1.5}^{x} u(x, t) = e^{2x}(1 - 2\sqrt{2}t), \quad 0 < t \leq 1, \quad -5 < x < 1,
$$

$$
u(-5, t) = e^{-10t}, \quad 0 < t \leq 1,
$$

$$
u(1, t) = e^{2t}, \quad 0 < t \leq 1,
$$

$$
u(x, 0) = 0, \quad -5 < x < 1.
$$

The exact solution to the above problem is $u(x, t) = te^{2x}$. Note for this test we have $u(-5, t) \approx 0$ and $u_x(-5, t) \approx 0$, and so, we may straightforwardly apply our numerical scheme to this test without the transformation used in Section 2.1. This problem is solved using a sequence of meshes $h_k = \Delta t_k = \frac{1}{5} \times 2^{-k}$ for $k = 0, 1, ..., 5$. And for each $k$, the following discrete maximum norm is computed:

$$
E_i = \max_{0 < j < N} \max_{0 < i < M} \left\{ \left| u(x_i, t_j) - U_i^j \right| \right\},
$$

where $U = (U_i^j)$ denotes the numerical solution. These computed errors, along with computed rates of convergence $\log_2(E_{k+1}/E_k)$, for $k = 0, 1, ..., 5$ are listed in Table 2.1, from which we see that the convergent rate of our method are of order $O(\Delta t^2 + h^2)$, coinciding with the truncation error bound established in Theorem 2.8. For comparison, we have also solved the problem using a combination of the Crank-Nicolson time-stepping
scheme and two popular existing spatial finite difference methods proposed respectively in [49] and [40]. These two existing methods are denoted as GL and L2 respectively. The computed errors $E_k^{GL}$'s and $E_k^{L2}$'s and the rates of convergence for GL and L2 are also listed in Table 2.1, from which it is clear that both of the existing methods are 1st-order accurate, one order lower than our method.

$$h = \Delta t = \frac{1}{5 \times 2^k}$$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h^{n+1}$</th>
<th>$E_k^{GL}$</th>
<th>$\log_2 \frac{E_k^{n+1}}{E_k^n}$</th>
<th>$E_k^{L2}$</th>
<th>$\log_2 \frac{E_k^{n+1}}{E_k^n}$</th>
<th>$E_k$</th>
<th>$\log_2 \frac{E_k^{n+1}}{E_k^n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.12244</td>
<td>0.17372</td>
<td>0.230950</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.06201</td>
<td>0.98</td>
<td>0.09873</td>
<td>0.82</td>
<td>0.049195</td>
<td>2.23</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.03120</td>
<td>0.99</td>
<td>0.05368</td>
<td>0.88</td>
<td>0.011224</td>
<td>2.13</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.01565</td>
<td>1.00</td>
<td>0.02829</td>
<td>0.92</td>
<td>0.000280</td>
<td>2.00</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.00784</td>
<td>1.00</td>
<td>0.01463</td>
<td>0.95</td>
<td>0.000070</td>
<td>2.00</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.00392</td>
<td>1.00</td>
<td>0.00748</td>
<td>0.97</td>
<td>0.000017</td>
<td>2.00</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.1: Computed Rates of Convergence for Example 2.1

Let us now consider the application of our method to FBS equations in the following two examples: European vanilla call and European vanilla put option. The system parameters for all these options are contained in Table 2.2.

<table>
<thead>
<tr>
<th>Parameters Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
</tr>
<tr>
<td>$\sigma$</td>
</tr>
<tr>
<td>$a$</td>
</tr>
<tr>
<td>$x_{\min}$</td>
</tr>
<tr>
<td>$x_{\max}$</td>
</tr>
<tr>
<td>$r$</td>
</tr>
<tr>
<td>$K$</td>
</tr>
<tr>
<td>$b$</td>
</tr>
<tr>
<td>$T$</td>
</tr>
</tbody>
</table>

Table 2.2: Market Parameters for Example 2-3

**Example 2.2.** [European Call Option] We consider the solution of (2.1) with the parameters listed in Table 2.2 and initial and boundary conditions are given as follows:

$$U(x, T) = \max(e^x - K, 0),$$

$$U(x_{\min}, t) = 0,$$

$$U(x_{\max}, t) = e^{x_{\max}} - K e^{-r(T-t)}.$$

To solve this problem, we choose the spatial mesh size $h = 0.02$ and time step $\Delta t = 1/52$. The numerical solution from our method for $\alpha = 1.5$ is plotted in Figure 2.2 against
Chapter 2. European Options

time t and the original independent variable stock price \( S = e^x \). From Figure 2.2, we see that the numerical method is very stable.

![Figure 2.2: Computed Value of European Call Option](image)

To see the influence of \( \alpha \) on the option price, we solve the problem for \( \alpha = 1.3, 1.5, 1.7 \) and 2, and plot in Figure 2.3 the difference between the values from the FBS equation (i.e., \( \alpha < 2 \)) and those from the BS equation (\( \alpha = 2 \)), denoted respectively \( C_{FBS}(x,t) \) and \( C_{BS}(x,t) \) at \( t = 0 \). From Figure 2.3, we see that the call price increases as \( \alpha \) decreases when \( S \) is greater than a critical value. This is because when \( \alpha \) is close to 1, the solution to the FBS equation exhibits the jump (or convection) nature, while when \( \alpha \) is close to 2, it is of mainly diffusion nature. As a result, an option on a stock of jump nature is more expensive than one of diffusion nature.
Chapter 2. European Options

Figure 2.3: Computed European Call Option Price Difference $C_{bs} - C_{fbs}$ for $\alpha = 1.3, 1.5$ and 1.7 at $t = 0$.

Example 2.3. [European Put Option] The market parameters are listed in Table 2.2, and the boundary and initial conditions are

$$U(x, T) = \max(K - e^x),$$
$$U(x_{\min}, t) = Ke^{-r(T-t)},$$
$$U(x_{\max}, t) = 0.$$

The problem has been solved using the same mesh as that for Example 2.2, and the solution for $\alpha = 1.5$ is depicted in Figure 2.4. To gauge the influence of $\alpha$ on the value of the option, we have also solved the problem for $\alpha = 1.3, 1.5, 1.7$ and 2. The difference between the value $P_{fBS}(x, t)$ from the FBS model and the value $P_{BS}(x, t)$ from the BS model at $t = 0$ is depicted in Figure 2.5 for each of the chosen values of $\alpha$.

From the figure, we see that the differences are qualitatively identical to those in Figure 2.3. In fact, it can be easily shown using the Put-Call Parity: $C_{fBS} - C_{BS} = P_{fBS} - P_{BS}$.
Our computation shows that

\[ \max_i \left| \left[ C_{fBS}(x_i, 0) - C_{BS}(x_i, 0) \right] - \left[ P_{fBS}(x_i, 0) - P_{BS}(x_i, 0) \right] \right| = 0.0672, \]

indicating that the Put-Call Parity is satisfied by our numerical results from the FBS equation.

![Figure 2.4: Computed Value of European Put Option](image)

**Figure 2.4:** Computed Value of European Put Option

### 2.6 Conclusion

In this section, we proposed a novel 2nd-order finite difference method for the fractional order differential operator, and 2nd-order finite difference method for the FBS equation governing European option pricing. The solvability and uniqueness of solution have been studied through its variational form by proving the continuity and coercivity of the FBS operator. We have proved the convergence of numerical method by showing that the discretization method is consistent, stable and monotone. In particular, we have shown that the truncation error of the scheme is of 2nd-order as opposed to the 1st-order truncation.
error for the existing numerical methods (eg. GL and L2) for the FBS equation. Fourier Analysis is applied to prove the stability of the discretization method. Two numerical experiments have been carried out to verify the theoretical findings. The result from the model fractional diffusion PDE shows that our method is 2nd-order accurate. The European option example also gives practically meaningful and correct results.

Figure 2.5: Computed European Put Option Price Difference $P_{FBS} - P_{BS}$ for $\alpha = 1.3, 1.5$ and $1.7$ at $t=0$. 
Chapter 3

American Options

In this Chapter, we propose a power penalty method to valuate American put option. Adding a power penalty term, the original differential complementarity problem becomes a nonlinear FBS equation. We establish the convergence of this power penalty method in the framework of the variational inequalities.

3.1 Fractional American Option Pricing Model

As mentioned in Section 1.3.2, it is known that the value $U$ of an American option satisfies the following linear complementarity problem [59, 60]:

$$
\mathcal{L} U \geq 0, \quad (3.1a)
$$

$$
U(x, t) - U^*(x) \geq 0, \quad (3.1b)
$$

$$
\mathcal{L} U(x, t) \cdot (U(x, t) - U^*(x)) = 0 \quad (3.1c)
$$

for $(x, t) \in (x_{\text{min}}, x_{\text{max}}) \times [0, T)$ with the the boundary and terminal conditions:

$$
U(x_{\text{min}}, t) = K, \quad U(x_{\text{max}}, t) = 0, \quad (3.2a)
$$

$$
U(x, T) = U^*(x), \quad (3.2b)
$$
where $\mathcal{L}$ is the FBS operator defined in (2.1). The payoff function of American vanilla put option is

$$U^* = \max[K - e^x, 0]. \quad (3.3)$$

Before considering the solvability of the above problem, we first transform (3.1) and (3.2a)-(3.2b) into one satisfying homogeneous Dirichlet boundary conditions by the same transforming method involving function (2.5) in Section 2.1. So we can get the following problem:

\begin{align*}
\mathcal{L}V(x,t) & \leq f(x,t), \quad (3.4a) \\
V(x,t) - V^*(x) & \leq 0, \quad (3.4b) \\
(\mathcal{L}V(x,t) - f(x,t)) \cdot (V(x,t) - V^*(x)) & = 0,
\end{align*}

satisfying the same boundary and payoff conditions:

\begin{align*}
V(x_{\text{min}},t) = 0 = V(x_{\text{max}},t), & \quad t \in [0,T), \quad (3.5a) \\
V(x,T) = V^*(x) := F(x) - U^*(x), & \quad x \in (x_{\text{min}},x_{\text{max}}), \quad (3.5b)
\end{align*}

where

$$f(x,t) = \mathcal{L}F = -\frac{(a-b)K}{e^{x_{\text{max}}} - e^{x_{\text{min}}}}e^x + rK \left(1 - \frac{e^x - e^{x_{\text{min}}}}{e^{x_{\text{max}}} - e^{x_{\text{min}}}}\right).$$

Using (3.3) and the definition of $F$, we can easily derive that

$$V^*(x) = \begin{cases} 
1 - \frac{K}{e^{x_{\text{max}}} - e^{x_{\text{min}}}} \left(1 - \frac{e^x - e^{x_{\text{min}}}}{e^{x_{\text{max}}} - e^{x_{\text{min}}}}\right), & x_{\text{min}} \leq x \leq \ln K, \\
1 - \frac{e^x - e^{x_{\text{min}}}}{e^{x_{\text{max}}} - e^{x_{\text{min}}}} K, & \ln K < x \leq e^{x_{\text{max}}}. 
\end{cases} \quad (3.6)$$

The problem (3.4) and the problem (3.1) are equivalent in the sense that their solutions are related by $V(x,t) = F(x) - U(x,t)$. The $\alpha$-th derivative of $V$ is defined the same as (2.8).
3.2 The Variational Formulation and Unique Solvability

In this subsection, we first formulate (3.4a) as a variational problem and then show that the variational problem has a unique solution.

Using the definition in Section 2.2, let

\[ K = \left\{ v(t) \in H^{\alpha/2}_0(I) : v(t) \leq V^*(t) \; \forall t \in [0, T) \right\}, \]

where \( V^* \) is the function defined in (3.6). It is easy to verify that \( K \) is a convex and closed subset of \( H^{\alpha/2}_0(I) \). Using the notation defined above, we pose the following problem:

**Problem 3.1.** Find \( V(t) \in K \), such that, for all \( v(t) \in K \),

\[
\left\langle -\frac{\partial V}{\partial t}, v - V \right\rangle + A(V, v - V) \geq (f, v - V),
\]

(3.7)

almost everywhere (a.e) \( t \in (0, T) \), satisfying terminal condition (3.5), where \( A(\cdot, \cdot) \) is a bilinear form defined by (2.9).

Using a standard argument, it can be easily shown that Problem 3.1 is the variational problem of (3.4). The bilinear form \( A(\cdot, \cdot) \) is coercive and continuous by the lemma 2.2.

**Theorem 3.2.** There exists a unique solution to Problem 3.1.

Similar to the proof of Theorem 2.3, this theorem is also a consequence of Lemma 2.2 and Theorem 1.33 in [25], in which the solvability for an abstract variational inequality problem is established. The coercivity and continuity of the bilinear form \( A(\cdot, \cdot) \) guarantee that Problem 3.1 is uniquely solvable.

**Theorem 3.3.** There exists a unique solution to equation (3.4) with boundary and payoff conditions (3.5a) – (3.5b).

3.3 Penalty Method and Convergence

Penalty methods have been used successfully for solving conventional constrained optimization problems and in recent years for continuous LCP involving 2nd-order differential operators. To our best knowledge, no penalty methods have been proposed or used for fractional order differential LCPs in the open literature. In this section, we will propose such a penalty method for (3.1a)–(3.1c). We then establish a convergence theory for the penalty method.
3.3.1 Penalty Method

Consider the following nonlinear equation

\[ \mathcal{L}V_\lambda(x,t) + \lambda[V_\lambda(x,t) - V^*(x)]_+^{1/k} = f(x,t), \quad (x,t) \in I \times (0,T), \]  

(3.8a)
satisfying the following boundary and terminal conditions:

\[ V_\lambda(x_{\min},t) = 0 = V_\lambda(x_{\max},t) \quad \text{and} \quad V_\lambda(x,T) = V^*(x), \]  

(3.8b)

where \([z]_+ = \max\{0,z\}\) for any function \(z\), and \(\lambda > 1\) and \(k > 0\) are parameters. The intuition of the above penalty equation is that the term \(\lambda[V_\lambda(x,t) - V^*(x)]_+^{1/k}\) in (3.8a) penalizes the part of \(V_\lambda\), which violates the constraint (3.4b) when \(\lambda\) or \(k\) is sufficiently large. The variational form of (3.8a) is as follows.

Problem 3.4. Find \(u_\lambda(t) \in H^{\alpha/2}(I)\) for \(t \in [0,T)\) a.e. satisfying (3.5), such that, for all \(v \in H^{\alpha/2}(I)\),

\[ \left\langle -\frac{\partial u_\lambda(t)}{\partial t}, v \right\rangle + A(u_\lambda(t), v) + \left(\lambda [u_\lambda(x,t) - V^*(x)]_+^{1/k}, v\right) = (f(t), v) \]  

(3.9)

where \(A\) is the bilinear form defined in Problem 3.1.

Before further discussion, it is necessary to introduce the usual Banach space in space and time and its \(L^p\)-norm defined respectively by

\[ L^p(0,T;H^0_0(I)) := \{ v(\cdot,t) : v(\cdot,t) \in H^0_0(I) \text{ a.e. in } (0,T); \|v(\cdot,t)\|_\gamma \in L^p((0,T)) \} \]

and

\[ \|v\|_{L^p(0,T;H^0_0(I))} := \left( \int_0^T \|v(\cdot,t)\|^p \|_\gamma dt \right)^{1/p}, \]

where \(1 \leq p \leq \infty\) and \((H^0_0(I),\|\cdot\|_\gamma)\) is the Sobolev space-norm pair defined in the previous section. We now have the following theorem.

Theorem 3.5. Problem 3.4 has a unique solution.

Proof. To prove this theorem, it suffices to show that the nonlinear operator on the LHS of (3.8a) is strongly monotone and continuous.
For any \( v_1(\cdot, t), v_2(\cdot, t) \in H^{\alpha/2}_0(I) \), a.e in \((0, T)\) satisfying (3.5), we have, using integration by parts,

\[
\langle \mathcal{L}(v_1 - v_2), v_1 - v_2 \rangle + \lambda \left( [v_1 - V^*]^{1/k}_+ - [v_2 - V^*]^{1/k}_+ , v_1 - v_2 \right) \\
= \left\langle -\frac{\partial (v_1 - v_2)}{\partial t}, v_1 - v_2 \right\rangle + A(v_1 - v_2, v_1 - v_2) \\
+ \lambda \left( [v_1 - V^*]^{1/k}_+ - [v_2 - V^*]^{1/k}_+ , v_1 - v_2 \right).
\]

(3.10)

Since \( [v - V^*]^{1/k}_+ = (\max[v - V^*, 0])^{1/k} \) is non-decreasing in \( v \), we have

\[
\int_{x_{\min}}^{x_{\max}} \left( [v_1 - V^*]^{1/k}_+ - [v_2 - V^*]^{1/k}_+ \right) (v_1 - v_2) \, dx \geq 0.
\]

Let \( e(t) = v_1(t) - v_2(t) \). Integrating both sides of (3.10) from 0 to \( T \) and using the above inequality and (2.10a) , we have

\[
\int_0^T \left\langle \mathcal{L}(e(\tau)), e(\tau) \right\rangle + \lambda \left( [v_1 - V^*]^{1/k}_+ - [v_2 - V^*]^{1/k}_+ , e(\tau) \right) \, d\tau \\
\geq \int_0^T \left\langle -\frac{\partial e(t)}{\partial t}, e(\tau) \right\rangle \, d\tau + \int_0^T A(e(\tau), e(\tau)) \, d\tau \\
\geq \int_0^T \left\langle -\frac{\partial e(t)}{\partial t}, e(\tau) \right\rangle \, d\tau + C_1 \| e \|_{L^2(0,T;H^{\alpha/2}_0(I))}.
\]

(3.11)

For any \( t \in (0, T) \), integrating by parts gives

\[
\int_t^T \left\langle -\frac{\partial e(t)}{\partial t}, e(\tau) \right\rangle \, d\tau = (e(t), e(t)) - \int_t^T \left\langle -\frac{\partial e(t)}{\partial t}, e(\tau) \right\rangle \, d\tau,
\]

since \( e(T) = v_1(T) - v_2(T) = 0 \). From this, we get

\[
\int_t^T \left\langle -\frac{\partial e(t)}{\partial t}, e(\tau) \right\rangle \, d\tau = \frac{1}{2} (e(t), e(t)) \geq 0.
\]

(3.12)

Therefore, combining (3.12) and (3.11) we see that the operator on the RHS of (3.9) is strongly monotone.

Moreover, from Lemma (2.2) we see that \( A(v, w) \) is Lipschitz continuous in both \( v \) and \( w \). Also, it is obvious that \( (v - V^*)^{1/k}_+ , w \) is continuous in both \( v \) and \( w \). Therefore, Problem 3.4 is uniquely solvable by the standard result in [25, Page 37].
3.3.2 Convergence

We now show that the solution to Problem 3.4 converges to that of (3.4) with the order $O(\lambda^{-k/2})$ in a proper norm. We first investigate the error bound for $[V_\lambda - V^*]_+$ in proper norms.

**Lemma 3.6.** Let $V_\lambda$ be the solution to Problem 3.4 and assume that $V_\lambda \in L^p(I)$, where $p = 1 + 1/k$. Then there exists a positive constant $C$, independent of $V_\lambda$ and $\lambda$, such that

$$
\| [V_\lambda - V^*]_+ \|_{L^p(I)} \leq C \lambda^{-k/2}.
$$

**Proof.** Let $C$ is a generic positive constant, independent of $V_\lambda$ and $\lambda$. For notation simplicity, we put $\phi(x,t) = [V_\lambda(x,t) - V^*(x)]_+$. It is easy to see that $\phi(\cdot , t) \in H^{\alpha/2}_0(I)$ for $t \in (0, T)$ a.e. Thus, setting $v = \phi$ in (3.9), we have

$$
\left\langle -\frac{\partial}{\partial t} V_\lambda, \phi \right\rangle + A(V_\lambda, \phi) + \lambda \left\langle \phi^{1/k}, \phi \right\rangle = (f, \phi) \quad \text{a.e. in } (0, T).
$$

Taking $-\left\langle \frac{\partial V^*}{\partial t}, \phi \right\rangle + A(V^*, \phi)$ away from both sides of the above equality gives

$$
\left\langle -\frac{\partial}{\partial t} (V_\lambda - V^*), \phi \right\rangle + A(V_\lambda - V^*, \phi) + \lambda \left\langle \phi^{1/k}, \phi \right\rangle = (f, \phi) + \left\langle \frac{\partial V^*}{\partial t}, \phi \right\rangle - A(V^*, \phi),
$$

or

$$
\left\langle -\frac{\partial \phi}{\partial t}, \phi \right\rangle + A(\phi, \phi) + \lambda \left\langle \phi^{1/k}, \phi \right\rangle = (f, \phi) - A(V^*, \phi),
$$

since $\phi = 0$ when $V_\lambda - V^* < 0$ and $\frac{\partial V^*}{\partial t} = 0$. Note $\phi(x,t) = [V_\lambda(x,t) - V^*(x)]_+ = 0$ by (3.8b). Integrating the above equality from $t$ to $T$ and using (3.12), (2.10a) and H"older Inequality, we get

$$
\int_t^T (f(\tau), \phi(\tau)) \, d\tau - \int_t^T A(V^*, \phi(\tau)) \, d\tau
\leq C \left( \int_t^T \|\phi(\tau)\|_{L^p(I)} \, d\tau \right)^{1/p} - \int_t^T A(V^*, \phi(\tau)) \, d\tau.
$$

(3.15)
Let $\beta = \alpha - 1 > 0$ in this chapter, from the definition of $A(\cdot, \cdot)$ in Problem 3.1, we see that
the integrand of the last term in (3.15) is
\[
- A(V^*, \phi(\tau)) = - \left\langle aV^* + b\frac{\partial^\beta V^*}{\partial x^\beta}, \frac{\partial \phi}{\partial x} \right\rangle - r(V^*, \phi). \tag{3.16}
\]
where $\frac{\partial^\beta V^*}{\partial x^\beta} = x_{\min} D_\beta^x V^* = - \infty D_\beta^x V^* - \infty D_{x_{\min}}^\beta V^*$. Since $- \infty D_{x_{\min}}^\beta = e^x$, so we can get
$x_{\min} D_\beta^x e^x = e^x - e^{\min}$. Thus from (3.6) we have
\[
\frac{\partial^\beta V^*(x)}{\partial x^\beta} = \begin{cases} (1 - \frac{K}{e^{\max} - e^{\min}})(e^x - e^{\min}) =: L_1 e^x > 0, & x_{\min} \leq x \leq \ln K, \\ -\frac{K}{e^{\max} - e^{\min}}(e^x - e^{\min}) =: L_2 e^x < 0, & \ln K \leq x \leq x_{\max}, \end{cases} \tag{3.17}
\]
since $0 < K/(e^{\max} - e^{\min}) < 1$. (Recall $0 < K < e^{\max} - e^{\min}$.) Note that $L_1 > 0, L_2 < 0$
and $L_1 - L_2 = 1$. Integrating by parts and using the fact that $V^*(x_{\min}) = 0 = V^*(x_{\max})$
for all $t \in (0, T)$, we obtain the following result from (3.17)
\[
- \int_{x_{\min}}^{x_{\max}} \frac{\partial^\beta V^*(x)}{\partial x^\beta} \cdot \frac{\partial \phi}{\partial x} dx = -L_1 \int_{x_{\min}}^{\ln K} e^x \frac{\partial \phi}{\partial x} dx - L_2 \int_{x_{\max}}^{\ln K} e^x \frac{\partial \phi}{\partial x} dx
\]
\[
= -(L_1 - L_2) K \phi(\ln K, t) + L_1 \int_{x_{\min}}^{\ln K} e^x \phi dx + L_2 \int_{x_{\max}}^{\ln K} e^x \phi dx
\]
\[
\leq C \int_{x_{\min}}^{x_{\max}} \phi dx. \tag{3.18}
\]
Similarly to the above, it is easy to show that
\[
- \left\langle aV^*, \frac{\partial \phi}{\partial x} \right\rangle - r(V^*, \phi) \leq C \int_{x_{\min}}^{x_{\max}} \phi dx. \tag{3.19}
\]
Integrating both sides of (3.16) from $t$ to $T$ and using (3.18) and (3.19) we have
\[
- \int_t^T A(V^*, \phi(\tau)) d\tau \leq C \int_t^T \int_{x_{\min}}^{x_{\max}} \phi(x, \tau) dx d\tau \leq C \left( \int_t^T \|\phi(\tau)\|_{L^p(I)}^p d\tau \right)^{1/p}.
\]
Therefore, replacing the last term in (3.15) by the above upper bound gives
\[
\frac{1}{2} (\phi(t), \phi(t)) + C \int_t^T \|\phi(\tau)\|_{\alpha/2}^2 d\tau + \lambda \int_t^T \|\phi(\tau)\|_{L^p(I)}^p d\tau \leq C \left( \int_t^T \|\phi(\tau)\|_{L^p(I)}^p d\tau \right)^{1/p} \tag{3.20}
\]
for all $t \in (0, T)$. This implies that

$$\lambda \int_t^T \| \phi(\tau) \|^p_{L^p(I)} d\tau \leq C \left( \int_t^T \| \phi(\tau) \|^p_{L^p(I)} d\tau \right)^{1/p},$$

or

$$\left( \int_t^T \| \phi(\tau) \|^p_{L^p(I)} d\tau \right)^{1-1/p} \leq C \lambda^{-1}.$$ 

Since $1 - 1/p = 1/(kp)$, it follows from the above estimate that

$$\left( \int_t^T \| \phi(\tau) \|^p_{L^p(I)} d\tau \right)^{1/p} \leq C \lambda^{-k}.$$ 

This is (3.13). Combining (3.20) and the above estimate yields

$$\frac{1}{2} (\phi(t), \phi(t)) + \int_t^T \| \phi(\tau) \|^2_{H^{\alpha/2}(I)} d\tau \leq \frac{C}{\lambda^k}.$$ 

Finally, the above inequality implies (3.14).

Using Lemma 3.6, we establish the main convergence result in the following theorem.

**Theorem 3.7.** Let $V$ and $V_\lambda$ be the solutions to Problems 3.1 and 3.4, respectively. If \( \frac{\partial V}{\partial t} \in L^{k+1}(\Omega) \), then there exists a constant $C > 0$, independent of $\lambda$, such that

$$\| V_\lambda - V \|_{L^\infty(0,T;L^2(I))} + \| V_\lambda - V \|_{L^2(0,T;H_0^{\alpha/2}(I))} \leq \frac{C}{\lambda^{k/2}},$$

where $k$ and $\lambda$ are the parameters used in (3.8a).

**Proof.** Let $\phi(\cdot, t) = [V_\lambda(\cdot, t) - V^*(\cdot)]_+$. Then, $V - V_\lambda$ can be decomposed as

$$V - V_\lambda = V - V^* + [V_\lambda - V^*]_- - [V_\lambda - V^*]_+ =: r_\lambda - \phi,$$

where $[z]_- = -\min\{z, 0\}$ for any $z$ and

$$r_\lambda = V - V^* + [V_\lambda - V^*]_-.$$ 

44
We first consider the estimation of $r_\lambda$. Setting $v = V - r_\lambda$ in (3.7) and $V = r_\lambda$ in (3.9), we have

$$\left\langle -\frac{\partial V}{\partial t}, r_\lambda \right\rangle + A(V, r_\lambda) \geq (f, r_\lambda),$$

$$\left\langle -\frac{\partial V}{\partial t}, r_\lambda \right\rangle + A(V, r_\lambda) + \lambda \left( \phi^{1/k}, r_\lambda \right) = (f, r_\lambda).$$

Adding up the above inequality and equality gives

$$\left\langle -\frac{\partial (V - V_\lambda)}{\partial t}, r_\lambda \right\rangle + A(V - V_\lambda, r_\lambda) + \lambda \left( \phi^{1/k}, r_\lambda \right) \geq 0.$$  

Note that

$$(\phi^\gamma, [V_\lambda - V^*]_-) = [V_\lambda - V^*]_+[V_\lambda - V^*]_- \equiv 0, \text{ for any } \gamma > 0,$$

using (3.23), we have

$$\left( \phi^{1/k}, r_\lambda \right) = \left( \phi^{1/k}, V - V^* + [V_\lambda - V^*]_- \right) = \left( \phi^{1/k}, V - V^* \right) \leq 0,$$

since $\phi \geq 0$ and $V - V^* \leq 0$. Therefore, (3.24) reduces to

$$\left\langle -\frac{\partial (V - V_\lambda)}{\partial t}, r_\lambda \right\rangle + A(V - V_\lambda, r_\lambda) \leq 0.$$ 

From (3.22), the inequality becomes

$$\left\langle -\frac{\partial r_\lambda}{\partial t}, r_\lambda \right\rangle + A(r_\lambda, r_\lambda) \leq \left\langle \frac{\partial \phi}{\partial t}, r_\lambda \right\rangle + A(\phi, r_\lambda).$$

By Cauchy Schwarz inequality we can have

$$(\phi(t), r_\lambda(t)) \leq \|\phi\|_{L^2(I)} \|r_\lambda\|_{L^2(I)}.$$ 

By the equivalence of the two norms $\|\cdot\|_{L^2(I)}$ and $\|\cdot\|_{L^2(0,T;L^2(I))}$, we can have $(\phi(t), r_\lambda(t)) \leq \|\phi\|_{L^2(0,T;L^2(I))} \|r_\lambda\|_{L^2(0,T;L^2(I))}$. Since $r_\lambda(x, T) = 0$, integrating both sides of the above
estimate from $t$ to $T$ and using the same argument for (3.12), we have

$$\frac{1}{2} (r_\lambda(t), r_\lambda(\tau)) + \int_t^T A (r_\lambda(\tau), r_\lambda(\tau)) \, d\tau$$

\[
\leq \int_t^T \left< -\frac{\partial \phi(\tau)}{\partial \tau}, r_\lambda \right> \, d\tau + \int_t^T A (\phi(\tau), r_\lambda(\tau)) \, d\tau
\]

\[
\leq (\phi(t), r_\lambda(t)) + \int_t^T \left< \phi(\tau), \frac{\partial r_\lambda(\tau)}{\partial \tau} \right> \, d\tau + \int_t^T A (\phi(\tau), r_\lambda(\tau)) \, d\tau
\]

\[
\leq \|\phi\|_{L^2(0,T;L^2(I))} \|r_\lambda\|_{L^2(0,T;L^2(I))} + C \|\phi\|_{L^2(0,T;H^{\alpha/2}(I))} \|r_\lambda\|_{L^2(0,T;H^{\alpha/2}(I))}
\]

\[
+ \int_t^T \left< \phi(\tau), \frac{\partial r_\lambda(\tau)}{\partial \tau} \right> \, d\tau,
\]

(3.26)

for all $t \in (0, T)$.

Using (3.25), (3.23), and (3.13), we estimate the last term in (3.26) as follows:

\[
\int_t^T \left< \phi(\tau), \frac{\partial r_\lambda(\tau)}{\partial \tau} \right> \, d\tau = \int_t^T \left< \phi(\tau), \frac{\partial V(\tau)}{\partial \tau} \right> \, d\tau \leq C \|\phi\|_{L^p(\Omega)} \left\| \frac{\partial V}{\partial t} \right\|_{L^q(\Omega)} \leq \frac{C}{\lambda^k},
\]

where $p = 1 + 1/k$ and $q = k + 1$, so that $1/p + 1/q = 1$. Substituting the above upper bound into (3.26) and using (2.10b), (2.10a) and (3.14), we obtain

\[
\left( \|r_\lambda\|_{L^\infty(0,T;L^2(I))} + \|r_\lambda\|_{L^2(0,T;H^{\alpha/2}(I))} \right)^2
\]

\[
\leq C \left( (1/2) \|r_\lambda\|_{L^\infty(0,T;L^2(I))}^2 + \|r_\lambda\|_{L^2(0,T;H^{\alpha/2}(I))}^2 \right)
\]

\[
\leq C \left[ \left( \|\phi\|_{L^2(0,T;L^2(I))} + \|\phi\|_{L^2(0,T;H^{\alpha/2}(I))} \right) \left( \|r_\lambda\|_{L^2(0,T;L^2(I))} + \|r_\lambda\|_{L^2(0,T;H^{\alpha/2}(I))} \right) + \lambda^{-k} \right]
\]

\[
\leq C \lambda^{-k/2} \left( \|r_\lambda\|_{L^2(0,T;L^2(I))} + \|r_\lambda\|_{L^2(0,T;H^{\alpha/2}(I))} \right) + \lambda^{-k}.
\]

This is of the form $\rho^2 \leq C (\rho \lambda^{-k/2} + \lambda^{-k})$ and it is easy to prove that $\rho \leq C \lambda^{-k/2}$ for a generic positive constant $C$, independent of $\lambda$. Therefore, we have

\[
\|r_\lambda\|_{L^\infty(0,T;L^2(I))} + \|r_\lambda\|_{L^2(0,T;H^{\alpha/2}(I))} \leq C \lambda^{-k/2}.
\]

(3.27)

Finally, using the triangular inequality, (3.22), (3.14) and (3.27), we can have

\[
\|V - V_\lambda\|_{L^\infty(0,T;L^2(I))} + \|V - V_\lambda\|_{L^2(0,T;H^{\alpha/2}(I))} \leq \left( \|r_\lambda\|_{L^\infty(0,T;L^2(I))} + \|r_\lambda\|_{L^2(0,T;H^{\alpha/2}(I))} \right)
\]

\[
+ \left( \|\phi\|_{L^\infty(0,T;L^2(I))} + \|\phi\|_{L^2(0,T;H^{\alpha/2}(I))} \right) \leq C \lambda^{-k/2}.
\]

This is (3.21). \qed
3.4 Discretization

Since the penalized FPDE cannot be solved analytically, it needs to be discretized in order to solve it numerically. Various discretization methods are available in the open literature. In this work, we apply the discretization scheme developed recently in [14] for (3.8a). For notation simplicity, we omit the subscript $\lambda$ in the rest of this section. We use the same mesh as in (2.11) and we use the same discretization of the $\alpha$-th derivative as in (2.19) in Chapter 2.

We now discretize the other terms in (3.8a). For a positive integer $N$, let $(0, T)$ be divided into $N$ subintervals with the mesh points $t_j = T - j\Delta t, j = 0, 1, ..., N$, where $\Delta t = T/N$. Thus $T = t_0 > t_1 > ... > t_j > ... > t_N = 0$. On this space-time mesh, we approximate (3.8a) by the following finite difference system:

\[
\frac{V_{j+1}^i - V_j^i}{\Delta t} + \frac{1}{2} \left( a \frac{V_{j+1}^{i+1} - V_{i-1}^j}{2h} - \frac{b}{\Gamma(2-\alpha)h^\alpha} \sum_{s=0}^{i+1} g_s V_{i+1-s}^j + r V_j^i + d_i(V_{j+1}^i) \right) \\
+ \frac{1}{2} \left( a \frac{V_{j+1}^i - V_{j-1}^i}{2h} - \frac{b}{\Gamma(2-\alpha)h^\alpha} \sum_{k=0}^{i+1} g_s V_{i+1-s}^j + r V_j^i + d_i(V_j^i) \right) = \frac{1}{2} \left( f_i^j + f_i^{j+1} \right)
\]

(3.28a)

for $i = 1, 2, ..., M - 1$ and $j = 1, 2, ..., N$ satisfying

\[
V_0^j = 0 = V_M^j, \quad V_i^0 = V^*(x_i), \quad i = 1, 2, ..., M - 1, \quad j = 1, 2, ..., N,
\]

(3.28b)

where $V_i^j$ denotes an approximation to $V(x_i, t_j)$, $d_i(V_i^j) = \lambda [V_i^j - V_i^1]^1/k$ and $f_i^j = f(x_i, t_j)$ for all feasible $(i, j)$. Clearly, (3.28a) is based on (2.19), the central differencing for $\partial V/\partial x$, the one-point quadrature rule for the 0th-order and the RHS terms and Crank-Nicolson time-stepping scheme.

We comment that for each $j = 1, 2, ..., N$, (3.28) is a nonlinear system in $(V_1^j, ..., V_{M-1}^j)^\top$ even when $k = 1$ in the penalty term. Also, the term $d_i(V_i^j)$ is non-smooth. In computation, a damped Newton’s algorithm and a smoothing technique can be used for solving (3.28).
Chapter 3. American Options

Then the above nonlinear system can be re-written as

\[
(I + \frac{1}{2} C) \vec{V}^{j+1} + \frac{1}{2} D(\vec{V}^{j+1}) = \frac{1}{2} \vec{f}^j + \frac{1}{2} \vec{f}^{j+1} + \left( I - \frac{1}{2} C \right) \vec{V}^j - \frac{1}{2} D(\vec{V}^j) \tag{3.29}
\]

where \( \vec{f}^{j+1} \) is the vector containing the contributions of the boundary conditions (3.28b) at time \( t_j \), \( \vec{V}^k = (V^k_1, V^k_2, \cdots, V^k_{M-1})^\top \) for \( k = j, j+1 \). \( V^j_i \) denotes the solution \( V(x_i, t_j) \) at node \( x_i \) and time level \( t_j \), \( j = 1, 2, \ldots, N-1 \), \( f_i^j = f(x_i, t_j) \). \( \vec{V}^j = (V^j_1, V^j_2, \cdots, V^j_{M-1})^\top \), \( D(\vec{V}^j) = (d_1(V^j_1), d_2(V^j_2), \ldots, d_{M-1}(V^j_{M-1}))^\top \) be a column vector.

Now we apply Newton’s method to (3.29), then we have the following results:

\[
\left( I + \frac{1}{2} C + \frac{1}{2} J_D(\omega^{l-1}) \right) \delta \omega^{l-1} = \vec{f}^j + \left( I - \frac{1}{2} C \right) \vec{V}^j - \frac{1}{2} D(\vec{V}^j) - \left( I + \frac{1}{2} C \right) \omega^{l-1} - \frac{1}{2} D(\omega^{l-1}), \tag{3.30}
\]

\[
\omega^l = \omega^{l-1} + \kappa \cdot \delta \omega^l
\]

for \( l = 1, 2, \ldots \), with \( C \) is defined in (2.27) and \( \omega^0 \) a given initial guess. \( J_D(\omega) \) denotes the Jacobian of the column vector \( D(\omega) \) and \( \kappa \in (0, 1] \) denotes a damping parameter. Then we choose \( \vec{V}^{j+1} = \lim_{l \to \infty} \omega^l \). \( \omega^l \) converges to \( \vec{V}^{j+1} \) quadratically if \( \omega^0 \) is sufficiently close to \( \vec{V}^{j+1} \). For detailed discussions on these in both infinite and finite dimensions, we refer to [28, 59].

Note that when \( V^j_i - V^*_i \to 0^+ \), \( d_i(V^j_i) \to \infty \), thus the Jacobian matrix is singular. To overcome this difficulty, a smooth function method in [59] is needed to deal with the non-Lipschitz continuous problem. They smooth out \( d_i(u^j_i) \) in the neighbourhood of \( [u^j_i - u^*_i]_+ = 0 \).

### 3.5 Convergence of Discretization for Linear Penalty Method

In this section, we only consider the convergence rate for the linear penalty method where \( k = 1 \). We show that the solution to (3.28) converges to the viscosity solution to (3.4) by proving that the numerical scheme proposed in the previous section is consistent, stable and monotone. We start this with the following theorem:
Chapter 3. American Options

**Theorem 3.8.** (Consistency) The finite difference scheme for (3.1), defined by (3.28), is consistent, with a truncation error of order $O(\Delta t^2 + h^2)$.

**Theorem 3.9.** (Stability) The finite difference scheme defined by (3.28) is unconditionally stable.

**Proof.** We use the discrete Fourier transform to prove the stability of the Crank-Nicolson method. Using $\mu$ and $\eta$ introduced in Subsection 2.3.2, we rewrite (3.28) as

$$d_i(V_i^k) = \lambda[V_i^k - V_i^*]_+ = \frac{\lambda}{2} \left[ \text{sign}(V_i^k - V_i^*) + 1 \right] (V_i^k - V_i^*) =: \rho_i^j(V_i^k - V_i^*)$$

where $\rho_i^j = 0$ or 1. Therefore, (3.28) can be written as

$$V_i^{j+1} - V_i^j + \frac{1}{2} \left[ \eta \left(V_{i+1}^j - V_{i-1}^j\right) + \mu \sum_{s=0}^{i+1} g_s V_{i-s+1}^{j+1} + (r + \lambda \rho_i^{j+1}) \Delta t V_i^{j+1} \right]$$

$$+ \frac{1}{2} \left[ \eta \left(V_{i+1}^j - V_{i-1}^j\right) + \mu \sum_{s=0}^{i+1} g_s V_{i-s+1}^j + (r + \lambda \rho_i^j) \Delta t V_i^j \right]$$

$$= \Delta t \left( \bar{f}_i^j + \bar{\rho}^j_i \lambda V_i^* \right), \tag{3.31}$$

where $\bar{\rho}^j_i = (\rho_i^j + \rho_i^{j+1})/2 = 0, 1/2$ or 1. Comparing (3.31) with (2.32) we see that (3.31) is in the same form as (2.32) with $r\Delta t$ replaced with $\Delta(r + \lambda \rho^k_i)$ for $k = j$ or $j + 1$ and $\Delta \bar{f}_i^j$ with $\Delta t \left( \bar{f}_i^j + \bar{\rho}^j_i \lambda V_i^* \right)$. All of these term are of the order $O(\Delta t)$. Thus, following the same analysis presented above for $\lambda = 0$, we have that the scheme is also stable when $\lambda > 0$. Therefore, we have proved the theorem.

We now show that the numerical scheme is monotone.

**Theorem 3.10.** (Monotonicity) The discretization scheme established in (3.28) is monotone when $\Delta t \leq \frac{2}{r}$.

**Proof.** When $\lambda > 0$, it is easy to see $d(V_i^k)$ is monotonically increasing in $V_i^k$ for any feasible $k$. Therefore, both $d(V_i^j)$ and $d(V_i^{j+1})$ monotone and thus the scheme is monotone. Therefore, the scheme is monotone.

Combining Theorems 3.8, 3.9 and 3.10, we have the following convergence result.
Chapter 3. American Options

**Theorem 3.11.** (Convergence) Let $V$ be the viscosity solution to (3.4) and $V_{h,\Delta t}$ be the solution to (3.28). Then, $V_{h,\Delta t}$ converges to $V$ as $(h, \Delta t) \to (0, 0)$.

Conventionally, Theorems 3.8 and 3.9 already imply the convergence of our numerical scheme, thus the proof is omitted here.

### 3.6 Numerical Results

To show the convergence rate and usefulness of the penalty and discretization methods, we apply these methods to two examples in this section. In the first example, we use a model LCP to demonstrate the convergence rate of our discretization method. We will then use a model American put option pricing problem to verify the theoretical rate of convergence of the penalty method obtained in Section 3.3.

#### 3.6.1 LCP Model

In this example, we demonstrate the second order convergent rate of this discretization method (3.28).

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} - 0D_x^{1.5}u(x,t) \geq f(x), & 0 < t < 1, \quad 0 < x < 2, \\
u(x,t) \geq g(x,t), \\
\left(\frac{\partial u(x,t)}{\partial t} - 0D_x^{1.5}u(x,t) - f(x)\right) \cdot (u(x,t) - g(x,t)) = 0.
\end{cases}
$$

with the boundary and initial conditions $u(0,t) = 0$, $u(2,t) = 8t$ for $0 < t < 1$, and $u(x,0) = 0$, for $0 \leq x \leq 2$, where $f(x) = x^3 - \frac{\Gamma(4)}{\Gamma(2.5)}x^{1.5}$ and $g(x,t) = xt$. The solution to the unconstrained problem is $x^3t$. The damping parameter $\kappa$ and the stopping criterion for the Newton’s method are chosen to be 0.02 and 0.0001 respectively. The space and time intervals $[0, 2]$ and $[0, 1]$ are divided into $M$ ($h = 2/M$) and $N$ ($\Delta t = 1/N$) subintervals.

To investigate the convergence rate of our discretization method, we fix the penalty parameter $\lambda = 1000$, and $k = 1$ for all meshes. Since there is no exact solution to this problem, we first solve this LCP on a large mesh where $M = 640$, $N = 320$, and use this numerical solution as our ‘exact’ or reference solution denoted as $u_r$. We solve this
Chapter 3. American Options

\[ M = 5 \times 2^l \quad N = 5 \times 2^{l-1} \]

<table>
<thead>
<tr>
<th>Error</th>
<th>1.0894e-1</th>
<th>2.5670e-2</th>
<th>5.9231e-3</th>
<th>1.3179e-3</th>
<th>2.6188e-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>log_2 \text{Ratio}</td>
<td>2.0854</td>
<td>2.1157</td>
<td>2.1682</td>
<td>2.3312</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>l</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
<tr>
<td>Error</td>
<td>1.0894e-1</td>
<td>2.5670e-2</td>
<td>5.9231e-3</td>
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<td>2.6188e-4</td>
</tr>
<tr>
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<td>2.0854</td>
<td>2.1157</td>
<td>2.1682</td>
<td>2.3312</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.1: Computed Rates of Convergence in \(dx\) and \(\Delta t\).

problem on a sequence of meshes \(M = 5 \times 2^l\) and \(N = 5 \times 2^{l-1}\) for \(l = 1, 2, 3, 4, 5\), and compute the corresponding approximations of errors defined as

\[
\|u_r - u_\lambda\|_{L^\infty(0,T;L^2(I))} + \|u_r - u_\lambda\|_{L^2(0,T;H^{\alpha/2}(I))}
\]

(3.32)
on these meshes. We also calculate logarithm based \(2\) of the ratio of the errors from two consecutive values of \(l\). Table 3.1 is a list of the computed errors for different values of \(M\) and \(N\). We can see that the convergence rates of the discretization are of second order.

3.6.2 American Option Pricing

In this section, we present some numerical results to verify the theoretical rates of convergence obtained in Section 3.3, by testing the convergence behaviours of the two penalty parameters \(\lambda\) and \(k\). The test problem is chosen to be the following one.

**Test Problem.** American put option with parameters: \(S_{\text{max}} = 100\), \(S_{\text{min}} = 0.1\), \(T = 1\), \(K = 50\), \(r = 0.05\), and \(\sigma = 0.25\).

To investigate the rates of convergence of the method in both \(\lambda\) and \(k\), we choose a fixed uniform mesh for the solution domain \((\ln(0.1), \ln(100)) \times (0,1)\) with \(M = 100\) and \(N = 104\). We also choose \(\alpha = 1.5\). Again the exact solution to this problem problem is unknown so we use the numerical solution with \(\lambda = 10^{10}\) and \(k = 1\) as our ‘exact’ or reference solution denoted as \(u_R\). Eq. (3.8a) corresponding to the problem are solved on the aforementioned uniform mesh (i.e., \(M = 100\), \(N = 104\)) for a sequence of values of \(\lambda\) when \(k\) is fixed. Approximations of the errors in the norm given in (3.32) on the mesh are computed and listed in Table 3.2 for chosen values of \(\lambda\) and \(k\). We also calculate logarithm based \(2\) of the ratio of the errors from two consecutive values of \(\lambda\) given in the first column of Table 3.2 for a fixed \(k\) and list them in Table 3.3. From (3.21), it is easily seen that the ratio is \(\log_2(2^{-nk/2}/2^{-(n+1)k/2}) = \log_2(2^{k/2}) = k/2\). From Table 3.3, we can see that the computed ratios behave like \(2^k\) rather than the theoretical ratio \(2^{k/2}\).
Chapter 3. American Options

\[ \lambda = 40 \times 2^{n-k} \]

<table>
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<tr>
<th>( \lambda = 40 \times 2^{n-k} )</th>
<th>( n = 1 )</th>
<th>( n = 2 )</th>
<th>( n = 3 )</th>
<th>( n = 4 )</th>
<th>( n = 5 )</th>
<th>( n = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 1 )</td>
<td>Error</td>
<td>8.44e-2</td>
<td>4.31e-2</td>
<td>2.18e-2</td>
<td>1.09e-2</td>
<td>5.39e-3</td>
</tr>
<tr>
<td>( \log_2 \text{Ratio} )</td>
<td>0.9693</td>
<td>0.9849</td>
<td>0.9924</td>
<td>0.9961</td>
<td>0.9963</td>
<td></td>
</tr>
<tr>
<td>( k = 2 )</td>
<td>Error</td>
<td>4.24e-2</td>
<td>1.09e-2</td>
<td>2.75e-3</td>
<td>6.91e-4</td>
<td>1.73e-4</td>
</tr>
<tr>
<td>( \log_2 \text{Ratio} )</td>
<td>1.9601</td>
<td>1.9863</td>
<td>1.9939</td>
<td>1.9985</td>
<td>1.9999</td>
<td></td>
</tr>
<tr>
<td>( k = 3 )</td>
<td>Error</td>
<td>7.89e-2</td>
<td>1.09e-2</td>
<td>1.38e-3</td>
<td>1.73e-4</td>
<td>2.17e-5</td>
</tr>
<tr>
<td>( \log_2 \text{Ratio} )</td>
<td>2.8610</td>
<td>2.9746</td>
<td>2.9947</td>
<td>3.0002</td>
<td>3.0051</td>
<td></td>
</tr>
<tr>
<td>( k = 4 )</td>
<td>Error</td>
<td>2.82e-1</td>
<td>4.11e-2</td>
<td>2.75e-3</td>
<td>1.74e-4</td>
<td>1.09e-5</td>
</tr>
<tr>
<td>( \log_2 \text{Ratio} )</td>
<td>2.7815</td>
<td>3.8986</td>
<td>3.9853</td>
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<tr>
<td>( k = 5 )</td>
<td>Error</td>
<td>5.73e-1</td>
<td>2.53e-2</td>
<td>2.11e-2</td>
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<td>( \log_2 \text{Ratio} )</td>
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<td>4.9883</td>
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</tr>
</tbody>
</table>

**Table 3.2:** Convergence behaviour with increasing \( \lambda \).

\[
\begin{array}{cccccc}
| \lambda = 5 | Errors & 2.92e-1 & 1.50e-1 & 7.89e-2 & 4.11e-2 & 2.11e-2 \\
| & Ratio & 1.9437 & 1.9066 & 1.9218 & 1.9459 |
\end{array}
\]

\[
\begin{array}{cccccc}
| \lambda = 10 | Errors & 1.61e-1 & 4.24e-2 & 1.09e-2 & 2.75e-3 & 6.95e-4 \\
| & Ratio & 3.8050 & 3.9062 & 3.9453 & 3.9596 |
\end{array}
\]

\[
\begin{array}{cccccc}
| \lambda = 20 | Errors & 8.44e-2 & 1.09e-2 & 1.38e-3 & 1.74e-4 & 2.18e-5 \\
| & Ratio & 7.7412 & 7.8912 & 7.9495 & 7.9774 |
\end{array}
\]

\[
\begin{array}{cccccc}
| \lambda = 40 | Errors & 4.31e-2 & 2.75e-3 & 1.73e-4 & 1.09e-5 & 6.86e-7 \\
| & Ratio & 15.6657 & 15.8740 & 15.9678 & 15.8167 |
\end{array}
\]

**Table 3.3:** Convergence behaviour with increasing \( k \).

indicating that the rate of convergence behaves like \( \lambda^{-k} \). In fact, it has been proved in [28, 58], using the facts that all the norms in finite dimensions are equivalent, the power penalty method for a nonlinear complementarity problem in finite dimensions satisfying a strong monotone condition has the convergence rate \( O(\lambda^{-k}) \). However, the convergence rate in finite dimensions is not uniform in the dimensionality. Since norms on a infinite-dimensional space are usually not equivalent, we are unable to achieve the \( O(\lambda^{-k}) \)-rate of convergence as in finite dimensions.

We now investigate computationally the rates of convergence of the method in \( k \) for a fixed \( \lambda \). Theoretically, from (3.21) we see the ratio of the errors in the solutions from \( k \) and \( k + 1 \) equals \( O(\lambda^{k+1}/\lambda^{k}) = O(\lambda) \). The computed results for different values of \( k \) and \( \lambda \) are listed in Table 3.3. From Table 3.3, we see that the ratios of the errors from two consecutive values of \( k \) for a fixed \( \lambda \) are almost constants.

We have also repeated the above numerical experiments for \( \alpha = 1.3, 1.7 \) and found
that the computed convergent rates are the same as the corresponding ones for $\alpha = 1.5$. Thus, the convergent rates of the penalty method does not depend on the fractional order $\alpha$.

To further demonstrate the performance of our method, we plot in Figure 3.1 the solution $V$, the Greek $\Delta = \partial V / \partial x$ and $V - V^*$ in the original independent variables $(S, t)$ for $\alpha = 1.5$ computed on the mesh used above using $k = 4$, $\lambda = 100$. From this figure, it can been seen that the numerical solutions are qualitatively very good and the bound constraint is always satisfied. In particularly, magnitude of $V - V^*$ matches that in [59] obtained from a standard American put option well.

![Graph of American put option V](image)

The value of American put option $V$

To see the influence of $\alpha$ on the option price, we solve the problem for $\alpha = 1.3, 1.5, 1.7$, and plot in Figure 3.2 the differences between the numerical solutions of the FBS equation and the standard BS equation (i.e., $\alpha = 2$) $P_{FBS} - P_{BS}$ at $t = 0$. From Figure 3.2, we see that the value of the American option is a decreasing function of $\alpha$ when $S$ is greater than a critical value, as observed in the Examples 2.2 and 2.3.

this American option with $\alpha = 1.5$ and the lower bound $U^*$ is plotted in, Figure 3.3. To see the difference between the American and European put options, we have also plot the difference between the computed European option value from Example 3 and $U^*$ in Figure 3.3. From the figure, we see that the American option is more expensive than
Figure 3.1: Computed $V$, $\Delta$ and $V - V^*$ for $\alpha = 1.5$ using $k = 4$ and $\lambda = 100$.

Figure 3.2: Computed Price Difference $P_{FBS} - P_{BS}$ for $\alpha = 1.3, 1.5$ and 1.7.
its European counterpart. Also, it can be seen that the value of the American option is bounded below by $U^*$, while the value of the European option falls below $U^*$ in a sub-region of the solution domain.

![Figure 3.3: Differences between Option prices and the Lower Bound: American option (upper), European option (lower)](image)

We also plot the prices of the American and European put options at the cross-section $t = 0$ in Figure 3.3, in which the lower bound $U^*$ is also displayed. From the figure, we see that the American option is more expensive than the European option. Also, the value of the American option touches the lower bound in the region from 0 to the point in the optimal exercise curve.

### 3.7 Conclusion

In this Chapter, we proposed and analyzed a power penalty method for the numerical solution of the fractional Black-Scholes equation governing American option pricing. We have shown that the existence and uniqueness of solution, and the solution from the penalty method converges to the solution of FBS equation at an arbitrary rate or order $O(\lambda^{-k/2})$. A discretization method is proposed for the penalized equation. Numerical results were presented to demonstrate the exponential rates of convergence of the penalty method and second order convergence rate of the discretization scheme.
Figure 3.4: Differences between Option Prices and the Lower Bound at $t = 0$
Chapter 4

European Two-Asset Options

In this Chapter, we develop a 2nd-order numerical scheme for a two dimensional fractional Black-Scholes equation arising in pricing options of European type on two assets under two independent geometric Lévy processes. We show that the continuous problem is uniquely solvable by the variational formulation. We prove the convergence of the numerical solution to the exact one by the consistency, stability and monotonicity of the discretization method. We also proposed an alternating-direction implicit method similar to Peaceman-Rachford type to increase the computing efficiency. Numerical results show that the numerical method is of 2nd-order accuracy under discrete maximum norm and the ADI method which can yield the same solution as the exact Crank-Nicolson method has much higher computational efficiency.

4.1 FBS Model for Two-Asset Options

The two-dimensional (2D) FBS model assumes that the two underlying assets follow two independent geometric Lévy processes.

As mentioned in the previous section, it is shown in [9] that the value of an option whose price follows a geometric Lévy process is governed by a one-dimensional FBS equation. Under the same assumptions as in [9], it is easy to show that the value $V$ of an option written on two stocks, (eg. Rainbow or Basket Option) whose prices $S_1$ and $S_2$ follow two independent geometric Lévy processes (with zero correlation coefficient), is determined by
Chapter 4. European Two-Asset Options

the following 2D FBS equation:

\[- U_t + a_1 U_x + a_2 U_y - b_1 [-\infty D_x^\alpha U] - b_2 [-\infty D_y^\beta U] + r U = 0 \] (4.1a)

for \((x, y, t) \in (-\infty, \infty)^2 \times [0, T]\), where \(x = \ln S_1\), \(y = \ln S_2\), and \(-\infty D_x^\alpha U\) and \(-\infty D_y^\beta U\) denote respectively the \(\alpha\)-th and \(\beta\)-th Caputo derivatives of \(U\) in \(x\) and \(y\) for \(\alpha, \beta \in (1, 2)\), \(T > 0\) is the terminal time, \(r \geq 0\) is the risk-free rate, \(\sigma > 0\) is the volatility of the underlying asset price. The coefficients are

\[a_1 = -r - \frac{1}{2} \sigma^\alpha \sec \left(\frac{\alpha \pi}{2}\right), \quad b_1 = -\frac{1}{2} \sigma^\alpha \sec \left(\frac{\alpha \pi}{2}\right),\]

\[a_2 = -r - \frac{1}{2} \sigma^\beta \sec \left(\frac{\beta \pi}{2}\right), \quad b_2 = -\frac{1}{2} \sigma^\beta \sec \left(\frac{\beta \pi}{2}\right).\]

In computation, the infinite solution domain \((-\infty, \infty)^2\) needs to be truncated into a finite domain \(\Omega = (x_{\min}, x_{\max}) \times (y_{\min}, y_{\max})\), where \(x_{\min}, x_{\max}, y_{\min}\) and \(y_{\max}\) are four constants satisfying \(x_{\min}, y_{\min} << 0\) and \(x_{\max}, y_{\max} > 0\). On \(\partial \Omega\), the boundary of \(\Omega\), and \(t = T\), we impose the following boundary and initial conditions

\[U(x, y, t) = U_0(x, y, t), \quad (x, y) \in \partial \Omega, \quad t \in (0, T], \] (4.1b)

\[U(x, y, T) = U^*(x, y), \quad (x, y) \in \Omega \] (4.1c)

satisfying the compatibility conditions \(U_0(x, y, T) = U^*(x, y)\) when \((x, y) \in \partial \Omega\).

Note that under this transformation \(x = \ln S_1, y = \ln S_2\), we have

\[\lim_{x \to -\infty} U_x = \lim_{y \to -\infty} U_{S_1} e^x = 0, \quad \lim_{y \to -\infty} U_y = \lim_{y \to -\infty} U_{S_2} e^y = 0,\]

since \(U_{S_1}(S_1)\) and \(U_{S_2}(S_2)\) are bounded as \(S_1, S_2 \to 0^+\) in practice. Therefore, when \(x_{\min}, y_{\min} << 0\), The following conditions for (4.1a), up to a truncation error, are satisfied:

\[U_x(x, y, t) = 0, \quad x \leq x_{\min}, (y, t) \in (y_{\min}, y_{\max}) \times [0, T),\]

\[U_y(x, y, t) = 0, \quad y \leq y_{\min}, (x, t) \in (x_{\min}, x_{\max}) \times [0, T).\]

In (4.1a), \(U_0(t)\) and \(U^*\) are known functions depending on the types of options and the strike price \(K\) of the option.
In this Chapter, we investigate on three different Rainbow options: Call-on-Min, Put-on-Min and Basket option. A Call-on-Min option gives the holder the right to purchase the minimum asset at the strike price at maturity and thus

\[ U^*(x, y) = \min(e^x, e^y) - K, \quad U_0(x, y, t) = \min(e^x, e^y) - Ke^{-r(T-t)} \]

where \([z]_+ = \max(0, z)\) for any real number \(z\). Similarly, a Put-on-Min option gives the holder the right to sell the maximum of the asset at maturity and so we have

\[ U^*(x, y) = K - \min(e^x, e^y), \quad U_0(x, y, t) = Ke^{-r(T-t)} - \min(e^x, e^y) \]

Also, the Basket option, whose underlying is a weighted average of two assets with the same strike price \(K\), has the payoff and boundary conditions as follows:

\[ U^*(x, y) = (w_1e^x + w_2e^y) - K, \quad U_0(x, y, t) = (w_1e^x + w_2e^y) - Ke^{-r(T-t)} \]

where the weighted coefficients satisfy \(0 \leq w_1, w_2 \leq 1\) and \(w_1 + w_2 = 1\).

Clearly, for all the above cases, \(K\) should satisfy \(0 < K < \min(e^{x_{\text{max}}} - e^{x_{\text{min}}}, e^{y_{\text{max}}} - e^{y_{\text{min}}})\). In what follows, we will refer to the operator \(L\) on the LHS of (4.1a) as the 2D FBS operator in this Chapter.

While the numerical solution of FBS equations arising in pricing an option written on single risky asset has been discussed in various existing works, to our best knowledge, there are no numerical methods for multi-dimensional FBS equations governing the valuation of options on multiple risky assets. In this paper we propose a 2nd-order numerical scheme for a 2D FBS equation arising in pricing options written on two assets, based on a combination of Crank-Nicolson method for the time derivative and a 2nd-order discretization scheme for the fractional partial derivative developed by us in [14] for a 1D fractional Black-Scholes equation.

### 4.2 Variational Formulation and Unique Solvability

We first reformulate (4.1a) as a variational problem, and then show that the variational problem has a unique solution.
Chapter 4. European Two-Asset Options

Before starting this discussion, we introduce some function spaces. For any \( \zeta = [\zeta_1, \zeta_2] \) and \( \zeta_1, \zeta_2 \in (0, 1] \), we let

\[
H^\zeta(\mathbb{R}^2) := \left\{ v : v, -\infty D_x^{\zeta_1}v \text{ and } -\infty D_y^{\zeta_2}v \in L^2(\mathbb{R}^2) \right\}.
\]

\(|\cdot|_\zeta\) and \(\|\cdot\|_\zeta\) are two functionals defined respectively as

\[
|u|_{\zeta_1} = \|\infty D_x^{\zeta_1}u\|_{L^2(\Omega)} \quad \text{and} \quad |u|_{\zeta_2} = \|\infty D_y^{\zeta_2}u\|_{L^2(\Omega)},
\]

\[
\|u\|_{\zeta_1} = \left( \|u\|_{L^2(\Omega)}^2 + |u|_{\zeta_1}^2 \right)^{1/2} \quad \text{and} \quad \|u\|_{\zeta_2} = \left( \|u\|_{L^2(\Omega)}^2 + |u|_{\zeta_2}^2 \right)^{1/2},
\]

\[
\|u\|_{\zeta} = |u|_{\zeta_1} + |u|_{\zeta_2}.
\]

for any \( u \in H^\zeta(\mathbb{R}^2) \). Then it is easy to show that \(|\cdot|_\zeta\) and \(\|\cdot\|_\zeta\) are semi-norm and norm on \(H^\zeta(\mathbb{R}^2)\) respectively. Based on the results from [19], we can show that \(H^\zeta(\mathbb{R}^2)\) equipped with \(\|\cdot\|_\zeta\) is a Sobolev space. We also define the Sobolev space of functions having a support on the finite interval \(\Omega = (x_{\min}, x_{\max}) \times (y_{\min}, y_{\max})\) given by

\[
H^\zeta_0(\Omega) = \left\{ v : v \in H^\zeta(\Omega), \ v|_{\partial\Omega} = 0 \right\},
\]

where \(x_{\min}D_x^{\zeta_1}u\) and \(y_{\min}D_y^{\zeta_2}u\) are defined in (2.8).

We now transform (4.1a) into the conservative form as follows:

\[
LU = -U_t - \nabla \cdot (aU + B\nabla^{(\gamma-1)}U) + rU,
\]

where

\[
a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.
\]

Let \(\gamma = [\alpha, \beta]\) and we define

\[
\nabla^{(\gamma-1)}U = \left[ \frac{\partial^{\gamma-1}U}{\partial x^{\alpha-1}}, \frac{\partial^{\gamma-1}U}{\partial y^{\beta-1}} \right]^T,
\]

where \(\frac{\partial^{\gamma-1}U}{\partial x^{\alpha-1}} = -\infty D_x^{\alpha_1}U\), and \(\frac{\partial^{\gamma-1}U}{\partial y^{\beta-1}} = -\infty D_y^{\beta_1}U\). Let \(U_0(x, y, t) \in H^\gamma(\Omega)\) for every \(t\), such that \(U_0\) satisfied the boundary conditions given in (4.1b) and \(\nabla U_0\) is continuous on
Chapter 4. European Two-Asset Options

Ω. Then we introduce a new function

\[ V(x, y, t) = U_0(x, y, t) - U(x, y, t). \]  \hfill (4.2)

Taking \( \mathcal{L}U_0 \) away from both sides of the (4.1a) and translating \( U \) into the new variable \( V \), it is easy to get that

\[ \mathcal{L}V = f, \]  \hfill (4.3a)

where

\[ \mathcal{L}V = -V_t - \nabla \cdot (B\nabla^{(\gamma-1)}V + aV) + rV, \]

and

\[ f(x, y, t) = \mathcal{L}V_0. \]

The boundary conditions and terminal condition now becomes

\[ V_0(x, y, t) = 0, \quad t \in [0, T] \quad \text{and} \quad (x, y) \in \partial \Omega, \]  \hfill (4.3b)

\[ V^*(x, y) = U_0(x, y, t) - U(x, y)^* \quad (x, y) \in \Omega, \]  \hfill (4.3c)

where \( U(x, y)^* \) is \( U(x, y, T) \). Using the notation defined above, we pose the following problem:

**Problem 4.1.** Find \( u(t) \in H^{\gamma/2}_0(\Omega) \), such that, for all \( v \in H^{\gamma/2}_0(\Omega) \),

\[ \left\langle -\frac{\partial u(t)}{\partial t}, v \right\rangle + A(u(t), v) = (f(t), v), \]

almost everywhere (a.e) in \((0, T)\) satisfying terminal condition (4.3c) a.e. in \((x_{\text{min}}, x_{\text{max}}) \times (y_{\text{min}}, y_{\text{max}})\), where \( A(\cdot, \cdot) \) is a bilinear form defined by

\[ A(u, v) = a \langle \nabla u, v \rangle + B \left\langle \nabla^{(\gamma-1)}u, \nabla v \right\rangle + r(u, v), \quad u, v \in H^{\gamma/2}_0(\Omega). \]  \hfill (4.4)

It is easy to verify that Problem 4.1 is the variational problem of (4.3a)–(4.3c) (cf. [19]). From lemma 2.2, we can prove that the bilinear form \( A(\cdot, \cdot) \) is coercive and continuous, as given in the following lemma:
Chapter 4. European Two-Asset Options

Lemma 4.2. There exist two positive constants $C_1^*$ and $C_2^*$, such that for any $v, w \in H_0^{\gamma/2}(\Omega),$

\[
A(v, v) \geq C_1^* \|v\|^2_{\gamma/2}, \quad (4.5)
\]
\[
A(v, w) \leq C_2^* \|v\|_{\gamma/2} \|w\|_{\gamma/2}, \quad (4.6)
\]

for $t \in (0, T)$ a.e..

Proof. Using Lemma 2.2 and Cauchy-Schwarz inequality, for $u, v \in H_0^{\gamma/2}(\Omega),$

\[
A(v, v) = a_1 \left\langle \frac{\partial v}{\partial x}, v \right\rangle + b_1 \left\langle x_{\min} D_x^{a-1} v, \frac{\partial v}{\partial x} \right\rangle + a_2 \left\langle \frac{\partial v}{\partial y}, v \right\rangle + b_2 \left\langle y_{\min} D_y^{\beta-1} v, \frac{\partial v}{\partial y} \right\rangle + r(v, v)
\]
\[
\geq C_1 \|v\|_{\alpha/2}^2 + C_2 \|v\|_{\beta/2}^2
\]
\[
\geq C^* (\|v\|_{\alpha/2} + \|v\|_{\beta/2})^2
\]
\[
\geq C_1^* \|v\|_{\gamma/2}^2.
\]

Also using Lemma 2.2 and Cauchy-Schwarz inequality, we can have the following proof for the continuity:

\[
A(v, w) = a_1 \left\langle \frac{\partial v}{\partial x}, w \right\rangle + b_1 \left\langle x_{\min} D_x^{a-1} v, \frac{\partial w}{\partial x} \right\rangle + a_2 \left\langle \frac{\partial v}{\partial y}, w \right\rangle + b_2 \left\langle y_{\min} D_y^{\beta-1} v, \frac{\partial w}{\partial y} \right\rangle + r(v, w)
\]
\[
\leq C_1 \|v\|_{\alpha/2} \|w\|_{\alpha/2} + C_2 \|v\|_{\beta/2} \|w\|_{\beta/2}
\]
\[
\leq C_2^* \left(\|v\|_{\alpha/2}^2 + \|v\|_{\beta/2}^2\right)^{1/2} \left(\|w\|_{\alpha/2}^2 + \|w\|_{\beta/2}^2\right)^{1/2}
\]
\[
= C_2^* \|v\|_{\gamma/2}^2 \|w\|_{\gamma/2}^2.
\]

\[\square\]

Theorem 4.3. There exists a unique solution to Problem 4.1.

This theorem is a consequence of Lemma 2.1 and Theorem 1.33 in [25], in which the unique solvability for an abstract variational problem is established. The proof to Theorem 4.3 is omitted here.
4.3 Discretization

Let the interval \((x_{\min}, x_{\max})\) and \((y_{\min}, y_{\max})\) be divided into \(M_x\) and \(M_y\) sub-intervals respectively with mesh nodes:

\[
x_i = x_{\min} + i h_x, \quad i = 0, 1, ..., M_x;
\]
\[
y_j = y_{\min} + j h_y, \quad j = 0, 1, ..., M_y,
\]

where \(h_x = (x_{\max} - x_{\min})/M_x\) and \(h_y = (y_{\max} - y_{\min})/M_y\). The \(\alpha\)-th partial derivatives with respect to \(x\) and \(y\) are defined in (2.8), and can be approximated the same as (2.19).

For a positive integer \(N\), let \((0, T)\) be divided into \(N\) sub-intervals with the mesh points

\[
t_n = T - n \Delta t, \quad n = 0, 1, ..., N,
\]

where \(\Delta t = T/N\). Thus

\[
T = t_0 > t_1 > ... > t_j > ... > t_N = 0.
\]

We also define the following finite difference operators for the fractional derivatives in (4.1a):

\[
\delta_x^\alpha U_{i,j}^n = \frac{1}{h_x} \frac{\alpha}{\Gamma(2 - \alpha)} \sum_{k=0}^{i+1} g_k^\alpha U_{i-k+1,j}^n, \quad (4.7a)
\]
\[
\delta_y^\beta U_{i,j}^n = \frac{1}{h_y} \frac{\beta}{\Gamma(2 - \beta)} \sum_{k=0}^{j+1} g_k^\beta U_{i,j-k+1}^n, \quad (4.7b)
\]

where \(U_{k,l}\) denotes an approximation to \(U(x_k, y_l, t)\) for all feasible \((k, l)\). We also define the following central difference approximations to the first partial derivatives:

\[
\delta_x U_{i,j}^n = \frac{1}{2h_x} (U_{i+1,j}^n - U_{i-1,j}^n), \quad \delta_y U_{i,j}^n = \frac{1}{2h_y} (U_{i,j+1}^n - U_{i,j-1}^n). \quad (4.7c)
\]
Using Crank-Nicolson time stepping method and the finite differences defined in (4.7a)-(4.7c), we construct the following discretization scheme for (4.1a):

\[
\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} + \theta \left( a_1 \delta_x U_{i,j}^{n+1} - b_1 \delta_x U_{i,j}^n + a_2 \delta_y U_{i,j}^{n+1} - b_2 \delta_y U_{i,j}^n + r U_{i,j}^{n+1} \right) \\
+ (1 - \theta) \left( a_1 \delta_x U_{i,j}^n - b_1 \delta_x U_{i,j}^n + a_2 \delta_y U_{i,j}^n - b_2 \delta_y U_{i,j}^n + r U_{i,j}^n \right) = 0 \quad (4.8a)
\]

for \( i = 1, ..., M_x - 1, j = 1, ..., M_y - 1 \), and \( n = 0, ..., N - 1 \). When \( \theta = 0.5 \), it is the Crank-Nicolson method. The boundary and payoff conditions for this system are:

\[
U_{0,j}^n = U_0(x_0, y_j, t_n), \quad U_{M_x,j}^n = U_0(x_{M_x}, y_j, t_n), \\
U_{i,0}^n = U_0(x_i, y_0, t_n), \quad U_{i,M_y}^n = U_0(x_i, y_{M_y}, t_n), \quad (4.8b)
\]

for all \( i = 1, ..., M_x - 1 \), \( j = 1, ..., M_y - 1 \) and \( n = 0, ..., N - 1 \), and

\[
U_{i,j}^N = U^*(x_i, y_j), \quad (x, y) \in \Omega. \quad (4.8c)
\]

To rewrite (4.8a) into a matrix form, we let

\[
\mathbf{V}^n = [U_{1,1}^n, U_{1,2}^n, \ldots, U_{1,M_x-1}^n, U_{2,1}^n, \ldots, U_{2,M_x-1}^n, \ldots, U_{M_y-1,1}^n, \ldots, U_{M_y-1,M_x-1}^n]^T,
\]

be a \((M_x - 1) \times (M_y - 1)\) column vector at \( n \)-th time level. Rearranging (4.8a), we can have

\[
(\mathbf{I} + \theta \mathbf{M}) \mathbf{V}^{n+1} = (\mathbf{I} - (1 - \theta) \mathbf{M}) \mathbf{V}^n + \mathbf{r}^{n+1-\theta}. 
\]
where \( I \) is \((M_x - 1) \times (M_y - 1)\) identity. The matrix \( M \) is a block matrix which has \((M_y - 1) \times (M_y - 1)\) blocks, and the size of each block matrix is \((M_x - 1) \times (M_x - 1)\).

\[
M = \begin{bmatrix}
A + B_1 & B_0 & 0 & \cdots & \cdots & \cdots & 0 \\
B_2 & A + B_1 & B_0 & \ddots & & & \\
B_3 & B_2 & A + B_1 & B_0 & \ddots & & \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \\
B_{M_y-3} & \ddots & B_2 & A + B_1 & B_0 & 0 \\
B_{M_y-2} & B_3 & B_2 & A + B_1 & B_0 \\
B_{M_y-1} & \cdots & \cdots & \cdots & B_3 & B_2 & A + B_1 \\
\end{bmatrix}_{(M_y-1) \times (M_y-1)}
\]

where

\[
A_{ij} = \begin{cases}
\mu_x g_{0}^\alpha + \eta_x, & j = i + 1 \\
\mu_x g_{0}^\alpha + \frac{r}{2} \Delta t, & j = i \\
\mu_x g_{2}^\alpha - \eta_x, & j = i - 1 \\
\mu_x g_{k}^\alpha, & j = i - k + 1, \ k = 3,4,\ldots,i \\
0, & \text{otherwise},
\end{cases}
\]

\[
B_{j} = \begin{cases}
(\mu_y g_{0}^\beta + \eta_y)I_y, & j = 0 \\
(\mu_y g_{1}^\beta + \frac{r}{2} \Delta t)I_y, & j = 1 \\
(\mu_y g_{2}^\beta - \eta_x)I_y, & j = 2 \\
(\mu_y g_{j}^\beta)I_y, & j = 3,4,\ldots,M_y.
\end{cases}
\]

\( f^{n+1-\theta} = (1-\theta)f^n + \theta f^{n+1} \) is the contribution of the boundary conditions (4.8b), where \( f^n \) and \( f^{n+1} \) are two \((M_x - 1) \times (M_y - 1)\) column vectors, composed of the contributions of the boundary terms. When \( \theta = 0.5 \), this discretization method is an exact Crank-Nicolson method.
4.4 Convergence

In this section, we show that the solution to (4.8a) converges to the viscosity solution to (4.1a) by proving that the numerical scheme proposed in the previous section is consistent, stable and monotone.

We start this with the following theorem:

**Theorem 4.4.** (Consistency) The finite difference scheme for (4.8a) is consistent with a truncation error of order $O(\Delta t^2 + h_x^2 + h_y^2)$.

**Proof.** We have shown in Theorem 2.8, the truncation error for the $\alpha$-th derivative has $h^2$ accuracy. So the finite difference schemes in (4.7a) and (4.7b) have the 2nd-order truncation error $O(h_x^2)$ or $O(h_y^2)$ respectively. In (4.7c), the central differencing also has 2nd-order truncation error. It is also known that the Crank-Nicolson’s time-stepping scheme used in (4.8a) has the truncation error of order $O(\Delta t^2)$. Therefore, the discretization scheme (4.8a) is consistent and has an order $O(\Delta t^2 + h_x^2 + h_y^2)$ truncation error. □

**Theorem 4.5.** (Stability) The finite difference scheme defined by (4.8a) is unconditionally stable.

**Proof.** Similar as in Section 2.9, we use the discrete Fourier transform to prove the stability of the Crank-Nicolson method. Applying Fourier transform to (4.8a) we replace $U_{i,j}^k$ with $U_{i,j}^k e^{(i\xi_1 + j\xi_2)h}$ with $i = \sqrt{-1}$ for all admissible $i$, $j$ and $k = n, n+1$. Dividing both sides by $e^{(i\xi_1 h_x + j\xi_2 h_y) i}$ and rearranging the resulting equation, we get

$$U^{n+1} = \left[ \frac{1 - \frac{1}{2} \left[ \eta_x \left( e^{i\xi_1 h_x i} - e^{-i\xi_1 h_x i} \right) + \mu_x \sum_{k=0}^{i+1} + \eta_y \left( e^{i\xi_2 h_y i} - e^{-i\xi_2 h_y i} \right) + \mu_y \sum_{k=0}^{j+1} \right] \right] U^n \right] + \frac{\Delta t \bar{F}^n}{1 + \frac{1}{2} \left[ \eta_x \left( e^{i\xi_1 h_x i} - e^{-i\xi_1 h_x i} \right) + \mu_x \sum_{k=0}^{i+1} + \eta_y \left( e^{i\xi_2 h_y i} - e^{-i\xi_2 h_y i} \right) + \mu_y \sum_{k=0}^{j+1} \right]},$$

where $\xi_1 \in [-\pi/h_x, \pi/h_x]$, $\xi_2 \in [-\pi/h_y, \pi/h_y]$ and $\mu_x, \mu_y, \eta_x, \eta_y$ are defined in (4.18), (4.19). We rewrite (4.9) in the following forms:
\[ U^{n+1} = \frac{1 - [(A_1 + A_2) + (B_1 + B_2)i]}{1 + [(A_1 + A_2) + (B_1 + B_2)i]} U^n + \left| \tilde{F} \right| \frac{2\Delta t}{1 + [(A_1 + A_2) + (B_1 + B_2)i]}, \]

where

\[ A_1 = \mu_x \sum_{k=0}^{i+1} g_k^\alpha (1 - k)\xi_1 h_x + r \frac{\Delta t}{4}, \]
\[ B_1 = \eta_x \sin(\xi_1 h_x) + \mu_1 \sum_{k=0}^{i+1} g_k^\alpha \sin((1 - k)\xi_1 h_x), \]
\[ A_2 = \mu_y \sum_{k=0}^{j+1} g_k^\beta (1 - k)\xi_2 h_y + r \frac{\Delta t}{4}, \]
\[ B_2 = \eta_y \sin(\xi_2 h_y) + \mu_2 \sum_{k=0}^{j+1} g_k^\beta \sin((1 - k)\xi_2 h_y). \]

Taking magnitudes on both sides of the equation above, we have

\[ |U^{n+1}| = |U^n| \sqrt{\frac{(1 - A_1 - A_2)^2 + (B_1 + B_2)^2}{(1 + A_1 + A_2)^2 + (B_1 + B_2)^2}} \frac{\Delta t}{\sqrt{(1 + A_1 + A_2)^2 + (B_1 + B_2)^2}}. \]

(4.10)

The estimation of \( A_1 \) and \( A_2 \) is the same as the estimation of \( A \) in (2.37), thus we can get the following result:

\[ A_1 \geq 0, \quad A_2 \geq 0, \]

(4.11)

from which, we can get

\[ \left| 1 - [(A_1 + A_2) + (B_1 + B_2)i] \right| \leq 1 + C\Delta t \]

(4.12)

with \( C = 0 \), independent of \( h \) and \( \Delta t \). Since the magnitude of the multiplication factor is less than 1, the numerical scheme (4.8a) is unconditionally stable.

We now show that the numerical scheme is monotone.

**Theorem 4.6.** (Monotonicity) The discretization scheme established in (4.8a) is monotone when \( \Delta t \leq \frac{2}{r} \).

67
Proof. Let

\[ F_{i,j}^{n+1} \left( t_{n+1}^{i,j}, U_{n+1}^{i,j}, U_{n+1}^{i-1,j}, \ldots, U_{0,j}^{i,j}, U_{n+1}^{i,j+1}, U_{n+1}^{i,j-1}, \ldots, U_{i,0}^{i,j}, U_{i+1,j}^{n} \right) := \left( \frac{1}{2} \left( \mu_x g_1^0 + \mu_y g_2^0 + r \Delta t \right) \right) U_{i,j}^{n+1} + \frac{1}{2} \left( \eta_x + \mu_y g_2^0 \right) U_{i,j+1}^{n+1} + \frac{1}{2} \left( \eta_y + \mu_y g_2^0 \right) U_{i,j-1}^{n+1} + \frac{1}{2} \mu_y \sum_{k=3}^{j+1} g_k^0 U_{i,j-k+1}^{n+1} \]

We have already proved that \( \left( \sum_{k=0}^{i+1} g_k \right) - \frac{1}{2} g_1 > 0 \) in Theorem 2.10. We now use this result to prove the monotonicity of \( F_{i,j}^{n+1} \). When \( \Delta t \leq \frac{2}{\alpha} \), we have from the definition of \( F_{i,j}^{n+1} \) that, for any \( \varepsilon > 0 \) and feasible \( i \) and \( j \),

\[
F_{i,j}^{n+1} \left( U_{i,j}^{n+1} + \varepsilon, U_{i,j+1}^{n+1} + \varepsilon, \ldots, U_{i,j}^{0,1} + \varepsilon, U_{i,j+1}^{0,1} + \varepsilon, \ldots, U_{i,0}^{n+1} + \varepsilon \right) - F_{i,j}^{n+1} \left( U_{i,j}^{n} + \varepsilon, U_{i,j+1}^{n}, \ldots, U_{i,j}^{0} + \varepsilon, U_{i,j+1}^{0}, \ldots, U_{i,0}^{n} + \varepsilon \right)
\]

\[
= \left( \frac{1}{2} \left( \mu_x g_1^0 + \mu_y g_1^0 \right) + r \Delta t \right) \varepsilon + \mu_x \left( g_0^0 + g_2^0 \right) \varepsilon + \mu_y \sum_{k=3}^{i+1} g_k^0 \varepsilon + \mu_y \sum_{k=3}^{j+1} g_k^0 \varepsilon
\]

\[
\leq F_{i,j}^{n+1} \left( U_{i,j}^{n+1} + \varepsilon, U_{i,j+1}^{n+1}, \ldots, U_{i,j}^{0,1} + \varepsilon, U_{i,j+1}^{0,1}, \ldots, U_{i,0}^{n+1} + \varepsilon \right) - F_{i,j}^{n+1} \left( U_{i,j}^{n} + \varepsilon, U_{i,j+1}^{n}, \ldots, U_{i,j}^{0} + \varepsilon, U_{i,j+1}^{0}, \ldots, U_{i,0}^{n} + \varepsilon \right)
\]

\[
= \left( \frac{1}{2} \left( \mu_x g_1^0 + \mu_y g_1^0 \right) + r \Delta t \right) \varepsilon + \mu_x \left( g_0^0 + g_2^0 \right) \varepsilon + \mu_y \sum_{k=0}^{i+1} g_k^0 \varepsilon - \left( \frac{1}{2} r \Delta t \right) \varepsilon
\]

68
Chapter 4. **European Two-Asset Options**

since \( \mu_x, \mu_y < 0 \). Furthermore, since \( g_\alpha^1 < 0 \) and \( g_\beta^1 < 0 \), we have

\[
F_{i,j}^{n+1}(U_{i,j}^{n+1} + \varepsilon, U_{i+1,j}^{n+1}, U_{i,j+1}^{n+1}, \ldots, U_{0,j}^{n+1}, U_{i,j}^{n+1}, U_{i+1,j}^{n+1}, U_{i,j-1}^{n+1}, \ldots, U_{0,j}^{n}, U_{i,j}^{n}, U_{i+1,j}^{n}, U_{i,j-1}^{n}, \ldots, U_{0,j}^{n}, U_{i,j}^{n}, U_{i+1,j}^{n}, U_{i,j-1}^{n}, \ldots, U_{0,j}^{n}) \\
= F_{i,j}^{n+1}(U_{i+1,j}^{n+1} + V_{i+1,j}^{n+1}, U_{i,j+1}^{n+1}, V_{i,j}^{n+1}, U_{i,j}^{n+1}, U_{i-1,j}^{n+1}, V_{i-1,j}^{n+1}, U_{i,j}^{n+1}, U_{i+1,j}^{n+1}, \ldots, U_{0,j}^{n}, U_{i,j}^{n}, U_{i+1,j}^{n}, U_{i,j-1}^{n}, \ldots, U_{0,j}^{n}, U_{i,j}^{n}, U_{i+1,j}^{n}, U_{i,j-1}^{n}, \ldots, U_{0,j}^{n}) + \\
\left( 1 + \frac{1}{2}(\mu_x g_\alpha^1 + \mu_y g_\beta^1 + r \Delta t) \right) \varepsilon
\]

Moreover, since \( g_{11}^\alpha < 0 \) and \( g_{11}^\beta < 0 \), we have

\[
F_{i,j}^{n+1}(U_{i,j}^{n+1} + \varepsilon, U_{i+1,j}^{n+1}, U_{i,j+1}^{n+1}, \ldots, U_{0,j}^{n+1}, U_{i,j}^{n+1}, U_{i+1,j}^{n+1}, U_{i,j-1}^{n+1}, \ldots, U_{0,j}^{n}, U_{i,j}^{n}, U_{i+1,j}^{n}, U_{i,j-1}^{n}, \ldots, U_{0,j}^{n}, U_{i,j}^{n}, U_{i+1,j}^{n}, U_{i,j-1}^{n}, \ldots, U_{0,j}^{n}) \\
= F_{i,j}^{n+1}(U_{i+1,j}^{n+1} + V_{i+1,j}^{n+1}, U_{i,j+1}^{n+1}, V_{i,j}^{n+1}, U_{i,j}^{n+1}, U_{i-1,j}^{n+1}, V_{i-1,j}^{n+1}, U_{i,j}^{n+1}, U_{i+1,j}^{n+1}, \ldots, U_{0,j}^{n}, U_{i,j}^{n}, U_{i+1,j}^{n}, U_{i,j-1}^{n}, \ldots, U_{0,j}^{n}, U_{i,j}^{n}, U_{i+1,j}^{n}, U_{i,j-1}^{n}, \ldots, U_{0,j}^{n}) + \\
\left( 1 + \frac{1}{2}(\mu_x g_\alpha^1 + \mu_y g_\beta^1 + r \Delta t) \right) \varepsilon
\]

Therefore, the scheme is monotone.

Combining Theorems 4.4, 4.5 and 4.6 , we have the following convergence result.

**Theorem 4.7.** (Convergence) Let \( U \) be the viscosity solution to (4.3a) – (4.3c) and \( U_{h_x,h_y,\Delta t} \) be the solution to (4.8a) – (4.8c). Then, \( U_{h_x,h_y,\Delta t} \) converges to \( U \) as \( (h_x, h_y, \Delta t) \to (0, 0) \).

These three Theorems 4.4 and 4.5 and 4.6 already conventionally imply the convergence of our numerical scheme. Barles and Souganidis showed in [1] that any finite difference scheme for a general nonlinear 2nd-order PDE which is locally consistent, stable and monotone generates a solution converging uniformly on a compact subset of \([0, T] \times \mathbb{R}\) to the unique viscosity solution of the PDE. In [15] and [16], Cont and Tankov extended this result to partial integro-differential equations (PIDEs). Since (2.1) is an PIDE, Theorem 2.11 is the consequence of the results established in [1, 15, 16].

### 4.5 Alternate Direction Implicit Method

In this section, we will then combine the discretization scheme developed in (2.19) with the ADI method similar to the one proposed in [50] and use the combination to solve Equation (4.1a). This numerical method is based on a ADI method of Peaceman-Rachford type. It has the merit that in every iteration of solving 2D problem, only two local 1D problems need to be solved. Thus, it reduces significantly computational costs for solving multi-dimensional problems.
Chapter 4. European Two-Asset Options

The Alternating Direction Implicit (ADI) method proposed by Peaceman and Rachford Jr. [50] in 1955 has been popular due to its computational effectiveness. In 1965, Mitchell and Fairweather proposed a scheme for wave equation which has second-order accuracy in time and fourth-order in space [21]. Since then, various improvements on the ADI methods have been obtained for multi-dimensional PDEs [22, 23]. In [45], Meerschaert et al. proposed a first order ADI method for a class of two-dimensional initial-boundary value fractional differential equations on a finite domain. In [55], the authors propose a second-order numerical method for a 2D FPDE based on a combination of an ADI scheme, a Crank-Nicolson time-stepping scheme and a Richardson extrapolation of a 1st-order spatial discretization method for the fractional derivatives. A compact ADI method for a two-dimensional time fractional diffusion equation is proposed in [57]. In this section, we propose a second order ADI method for the 2D FBS equation which is an advection-diffusion FPDE.

First, we rewrite Eq.(4.8a) in following form
\[
\left[ 1 + \frac{\Delta t}{2} \left( \delta_x - \delta_x^\alpha + \delta_y - \delta_y^\beta + r \right) \right] U_{i,j}^{n+1} = \left[ 1 - \frac{\Delta t}{2} \left( \delta_x - \delta_x^\alpha + \delta_y - \delta_y^\beta + r \right) \right] U_{i,j}^n. \tag{4.13}
\]

When \( \Delta t > 0 \) is sufficiently small, using (4.13) it is easy to show
\[
\begin{align*}
& \left( 1 + \frac{\Delta t}{2} \left( \delta_x - \delta_x^\alpha + \frac{1}{2} r \right) \right) \left( 1 + \frac{\Delta t}{2} \left( \delta_y - \delta_y^\beta + \frac{1}{2} r \right) \right) U_{i,j}^{n+1} \\
& = \left( 1 - \frac{\Delta t}{2} \left( \delta_x - \delta_x^\alpha + \frac{1}{2} r \right) \right) \left( 1 - \frac{\Delta t}{2} \left( \delta_y - \delta_y^\beta + \frac{1}{2} r \right) \right) U_{i,j}^n \\
& + \frac{\Delta t^2}{4} \left( \delta_x - \delta_x^\alpha + \frac{1}{2} r \right) \left( \delta_y - \delta_y^\beta + \frac{1}{2} r \right) \left( U_{i,j}^{n+1} - U_{i,j}^n \right), \tag{4.14}
\end{align*}
\]

for all feasible \((i,j,n)\). Omitting the term \( O(\Delta t^2) \) in (4.14), we define the following ADI scheme similar to that of the Peaceman-Rachford type:
\[
\begin{align*}
& \left( 1 + \frac{\Delta t}{2} \left( \delta_x - \delta_x^\alpha + \frac{1}{2} r \right) \right) \tilde{U}_{i,j} = \left( 1 - \frac{\Delta t}{2} \left( \delta_y - \delta_y^\beta + \frac{1}{2} r \right) \right) U_{i,j}^n, \tag{4.15a} \\
& \left( 1 + \frac{\Delta t}{2} \left( \delta_y - \delta_y^\beta + \frac{1}{2} r \right) \right) U_{i,j}^{n+1} = \left( 1 - \frac{\Delta t}{2} \left( \delta_x - \delta_x^\alpha + \frac{1}{2} r \right) \right) \tilde{U}_{i,j}. \tag{4.15b}
\end{align*}
\]

Note that \( \tilde{U}_{i,j} \) is an ‘intermediate’ solution in (4.15). To retrieve (4.13) (without the \( O(\Delta t^2) \) term), we can multiply (4.15a) and (4.15b) by \( 1 - \frac{\Delta t}{2} \left( \delta_x - \delta_x^\alpha + \frac{1}{2} r \right) \) and \( 1 + \frac{\Delta t}{2} \left( \delta_x - \delta_x^\alpha + \frac{1}{2} r \right) \) respectively, and use the resulting equations to eliminate \( \tilde{U}_{i,j} \).
Chapter 4. European Two-Asset Options

Note that (4.15) is a decoupled system, i.e., we can solve (4.15a) for
\[ \tilde{U}_j := \left( \tilde{U}_{1,j}, ..., \tilde{U}_{M_x-1,j} \right)^\top \]
first and then solve (4.15b) for approximation at \( t = t_{n+1} \). Thus, we need to define boundary conditions for (4.15a). To achieve this, we take both sides of (4.15b) from those of (4.15a) and re-arrange the resulting equation so that
\[ 2\tilde{U}_{i,j} = \left( 1 - \frac{\Delta t}{2} \left( \delta_y - \delta^\beta_y + \frac{1}{2}r \right) \right) U^n_{i,j} + \left( 1 + \frac{\Delta t}{2} \left( \delta_y - \delta^\beta_y + \frac{1}{2}r \right) \right) U^{n+1}_{i,j}. \]

Therefore, the boundary conditions for (4.15a) are
\[
\begin{align*}
\tilde{U}_{0,j} &= \frac{1}{2} \left[ \left( 1 - \frac{\Delta t}{2} \left( \delta_y - \delta^\beta_y + \frac{1}{2}r \right) \right) U^n_{0,j} + \left( 1 + \frac{\Delta t}{2} \left( \delta_y - \delta^\beta_y + \frac{1}{2}r \right) \right) U^{n+1}_{0,j} \right], \\
\tilde{U}_{M_x,j} &= \frac{1}{2} \left[ \left( 1 - \frac{\Delta t}{2} \left( \delta_y - \delta^\beta_y + \frac{1}{2}r \right) \right) U^n_{M_x,j} + \left( 1 + \frac{\Delta t}{2} \left( \delta_y - \delta^\beta_y + \frac{1}{2}r \right) \right) U^{n+1}_{M_x,j} \right],
\end{align*}
\]
for \( j = 1, 2, ..., M_y \). The following algorithm implements the ADI method.

Algorithm ADI.

Step 1. Initialize \( U^0_{i,j} \) for \( i = 1, 2, ..., M_x \) and \( j = 1, 2, ..., M_y \) using the payoff condition (4.1c).

For \( n = 0, 1, ..., N \), perform the following steps.

Step 2. For \( j = 1, 2, ..., M_y - 1 \), solve the following system with (4.16) for \( \tilde{U}_j \)
\[
\left( I + \frac{1}{2} C_x \right) \tilde{U}_j = \left( I - \frac{1}{2} C_y \right) U^n_{j} + \frac{1}{2} \tilde{p}^n_J,
\]
where \( \tilde{U}_j := (U^n_{1,j}, ..., U^n_{M_x-1,j})^\top \), \( \tilde{p}^n_j \) is the vector containing the contributions from the boundary conditions \( U^n_{0,j}, U^n_{M_x,j}, \tilde{U}_{0,j} \) and \( \tilde{U}_{M_x,j} \), and \( C_x = (c^x_{pq}) \) is an \((M_x-1) \times (M_x-1)\) matrix with
\[
c^x_{pq} = \begin{cases} 
\mu_x g^0_{q} + \eta_x, & q = p + 1, \\
\mu_x g^0_{q} + \frac{r}{2} \Delta t, & q = p, \\
\mu_x g^0_{q} - \eta_x, & q = p - 1, \\
\mu_x g^0_{q}, & q = p - k + 1, \quad k = 3, 4, ..., i \\
0, & \text{otherwise},
\end{cases}
\]
where
\[ \mu_x = -b_1 \frac{\Delta t}{\Gamma(2 - \alpha)h_x^\alpha}, \quad \eta_x = a_1 \frac{\Delta t}{2h_x}. \] (4.18)

The matrix \( C_y = (c_{pq}^y) \) in (4.17) is defined in the same way as that of \( C_x \) with \( \alpha \) replaced with \( \beta \) and \( \mu_x \) and \( \eta_x \) by
\[ \mu_y = -b_2 \frac{\Delta t}{\Gamma(2 - \alpha)h_y^\alpha}, \quad \eta_x = a_2 \frac{\Delta t}{2h_y}. \] (4.19)

respectively in the above definition for \( c_{pq}^y \).

Step 3. For \( i = 1, 2, ..., M_x - 1 \), solve the following system for \( \vec{U}^{n+1}_i := (U^{n+1}_{i,1}, ..., U^{n+1}_{i,M_y-1})^\top \):
\[
\left( I + \frac{1}{2}C_2 \right) \vec{U}^{n+1}_i = \left( I - \frac{1}{2}C_1 \right) \vec{U}^n_x + \frac{1}{2} \vec{f}^{n+1},
\]

where \( \vec{f}^{n+1} \) is a vector containing the contributions from the boundary values \( U^{n+1}_{i,0}, U^{n+1}_{i,M_y}, \vec{U}^n_{i,0} \)
and \( \vec{U}^n_{i,M_y} \).

4.6 Numerical Results

In this section, we first present two examples with known exact solutions to demonstrate the rate of convergence of our scheme, and then use this scheme to solve a 2D FBS equation for Rainbow Options and Basket Option pricing. In the first example, we compare the errors from the exact Crank-Nicolson (CN) solutions with the errors from the ADI solutions. The computational times shows that ADI has much higher efficiency than the exact CN method.

Example 4.1. The fractional convection-diffusion equation with homogeneous boundary conditions:
\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial u}{\partial y} - \frac{\partial^\beta u}{\partial y^\beta} = f(x, y, t).
\]
Chapter 4. European Two-Asset Options

where \( f(x,y,t) = x^3y^4 + \left(3x^2y^4 - \frac{\Gamma(4)}{\Gamma(4-\alpha)}x^3\alpha y^4 + 4x^3y^3 - \frac{\Gamma(5)}{\Gamma(5-\beta)}x^3\beta y^4 - x^3y^4\right) t, \)

\[
\frac{\partial^\alpha u}{\partial x^\alpha} = 0 \quad \text{and} \quad \frac{\partial^\beta u}{\partial y^\beta} = 0
\]

are defined in (2.4), with boundary conditions:

\[
u(x,0,t) = \nu(x,1,t) = 0, \quad x \in (0,1), \quad t \in (0,1], \]

\[
u(0,y,t) = \nu(1,y,t) = 0, \quad y \in (0,1), \quad t \in (0,1],
\]

\[
u(x,y,1) = x^3y^4, \quad (x,y) \in (0,1) \times (0,1).
\]

The exact solution to the above problem is \( u(x,t) = x^3y^4t. \) This problem is solved using a sequence of meshes \( h_x = h_y = \Delta t_k = h_k = \frac{1}{5} \times 2^{-k} \) for \( k = 0, 1, \ldots, 5. \) And for each \( k, \) the following discrete maximum norm is computed:

\[
E_i = \max_{0<n<N-1} \max_{1<i<M} \max_{1<j<M} \left\{ \left| u(x_i,y_j,t_j) - U_{ij}^n \right| \right\}, \quad (4.20)
\]

where \( U = (U_{ij}^n) \) denotes the numerical solution. These computed errors, along with computed rates of convergence \( \log_2(E_{k+1}/E_k), \) for \( k = 0, 1, \ldots, 5 \) are listed in Table 4.1, from which we see that the rate of convergence of our method is of order \( O(\Delta t^2 + h_x^2 + h_y^2). \) For comparison, we have also solved the problem using a combination of the Crank-Nicolson time-stepping scheme and Grünwald-Letnikov method in [49]. We used ADI method for both of these discretization schemes. The computed errors \( E_{k}^{GL}'s \) and the rates of convergence for GL are also listed in Table 4.1, from which it is clear that the existing method is 1st-order accurate, one order lower than our method.

<table>
<thead>
<tr>
<th>( h = \Delta t ) = ( \frac{1}{5 \times 2^k} )</th>
<th>( E_{k}^{GL} )</th>
<th>( \log_2 \frac{E_{k}^{GL}}{E_{k+1}} )</th>
<th>( E_k )</th>
<th>( \log_2 \frac{E_{k+1}}{E_k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = 0 )</td>
<td>1.9403e-02</td>
<td>6.6736e-03</td>
<td>( E_{k}^{GL} )</td>
<td>( \log_2 \frac{E_{k}^{GL}}{E_{k+1}} )</td>
</tr>
<tr>
<td>( k = 1 )</td>
<td>1.4090e-02</td>
<td>0.4616</td>
<td>2.0301e-03</td>
<td>1.7169</td>
</tr>
<tr>
<td>( k = 2 )</td>
<td>8.2916e-03</td>
<td>0.7650</td>
<td>5.4646e-04</td>
<td>1.8934</td>
</tr>
<tr>
<td>( k = 3 )</td>
<td>4.4631e-03</td>
<td>0.8936</td>
<td>1.3942e-04</td>
<td>1.9707</td>
</tr>
<tr>
<td>( k = 4 )</td>
<td>2.3158e-03</td>
<td>0.9466</td>
<td>3.5125e-05</td>
<td>1.9889</td>
</tr>
<tr>
<td>( k = 5 )</td>
<td>1.1801e-03</td>
<td>0.9726</td>
<td>8.8149e-06</td>
<td>1.9945</td>
</tr>
</tbody>
</table>

To demonstrate the efficiency of the ADI method, we compare errors of the numerical solution generated by the exact Crank-Nicolson method with those generated by the ADI method. We listed the elapsed times \( T_1 \) and \( T_2 \) (in second) for each computation with different mesh sizes and the ratio of \( T_1/T_2 \) for the same mesh size.
Table 4.2: Errors and Computational Cost of Example 4.1

<table>
<thead>
<tr>
<th>h = \Delta t = \frac{1}{5\times2^n}</th>
<th>E_{\text{exactCN}}</th>
<th>T_1 (s)</th>
<th>E_{\text{ADI}}</th>
<th>T_2 (s)</th>
<th>T_1/T_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>k = 0</td>
<td>5.5334e-03</td>
<td>0.0058</td>
<td>6.6736e-03</td>
<td>0.0141</td>
<td>1.9005</td>
</tr>
<tr>
<td>k = 1</td>
<td>1.6536e-03</td>
<td>0.0158</td>
<td>2.0301e-03</td>
<td>0.0291</td>
<td>1.8390</td>
</tr>
<tr>
<td>k = 2</td>
<td>4.3207e-04</td>
<td>0.0624</td>
<td>5.4646e-04</td>
<td>0.1649</td>
<td>2.6426</td>
</tr>
<tr>
<td>k = 3</td>
<td>1.0980e-04</td>
<td>0.4803</td>
<td>1.3942e-04</td>
<td>1.9574</td>
<td>2.6426</td>
</tr>
<tr>
<td>k = 4</td>
<td>2.7660e-05</td>
<td>3.2137</td>
<td>3.5125e-05</td>
<td>45.504</td>
<td>14.159</td>
</tr>
<tr>
<td>k = 5</td>
<td>6.9454e-06</td>
<td>28.950</td>
<td>8.8149e-06</td>
<td>1679.3</td>
<td>58.009</td>
</tr>
</tbody>
</table>

From Table 4.2, we can see the errors the exact CN solution is about 20% less than those of the ADI solution. This is because the ADI method causes extra truncation error when the two-dimensional differential operator is split into two one-dimensional operators. However, the ADI method still outperforms the exact Crank-Nicolson method because of its computational efficiency. The exact CN method demands a much larger computational cost than the ADI method. When the mesh size \( h = 1/160 \), the exact CN method runs about 60 times more computational work than the ADI method. Although the exact CN method provides slightly better solution, the computational cost is not affordable when the dimension of the system matrix increases. Therefore, the implementation of ADI method is necessary for solving a 2D FPDE.

Example 4.2. The fractional convection-diffusion equation with non-homogeneous boundary conditions:

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial u}{\partial y} - \frac{\partial^\beta u}{\partial y^\beta} = f(x, y, t),
\]

where \( f(x, y, t) = (e^{2x} + e^y) + (2e^{2x} - 2^\alpha e^{2x})t \),

\[
\frac{\partial^\alpha}{\partial x^\alpha}u = -\infty \quad \text{D}^\alpha_x u,
\]

and \( \frac{\partial^\beta}{\partial y^\beta}u = -\infty \quad \text{D}^\beta_y u \) are defined in (2.4), with boundary conditions:

\[
\begin{align*}
&u(x, 0, t) = u(x, 1, t) = 0, \quad y \in (0, 1), \quad t \in (0, 1], \\
&u(0, y, t) = u(1, y, t) = 0, \quad x \in (0, 1), \quad t \in (0, 1], \\
&u(x, y, 1) = e^{2x} + e^y, \quad (x, y) \in (0, 1) \times (0, 1].
\end{align*}
\]

The exact solution to the above problem is \( u(x, t) = (e^{2x} + e^y)t \). Since the lower bound of Caputo’s \( \alpha \)-th derivative is \( -\infty \), we solve this FPDE on the extended spatial domain \( (X_{\text{ext}}, 1) \times (Y_{\text{ext}}, 1) \) to minimize the truncation error on the left boundary. In this example, we choose \( X_{\text{ext}} = Y_{\text{ext}} = -31 \).
This problem is solved using the same sequence of meshes as in the Example 4.1 where \( x = X_{\text{ext}}, (X_{\text{ext}} + h), (X_{\text{ext}} + 2h), \ldots, -h, x_0, x_1, \ldots, x_{M_x} \) and \( y = Y_{\text{ext}}, (Y_{\text{ext}} + h), (Y_{\text{ext}} + 2h), \ldots, -h, y_0, y_1, \ldots, y_{M_y} \), where \( x_0 = y_0 = 0 \) and \( x_{M_x} = y_{M_y} = 1 \). We compare the norms defined in (4.20) of the errors of the numerical solutions with those from GL method again. From Table 4.3, we see that the convergent rate of our method is of order \( \mathcal{O}(\Delta t^2 + h_x^2 + h_y^2) \) while the GL method is only of the first order.

### Table 4.3: Errors of Example 4.2

<table>
<thead>
<tr>
<th>( k )</th>
<th>( E_k^{GL} )</th>
<th>( \log_2 \frac{E_k^{GL}}{E_k^{ex}} )</th>
<th>( E_k )</th>
<th>( \log_2 \frac{E_{k+1}}{E_k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6.0644e-01</td>
<td>3.0815e-01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.0751e-01</td>
<td>0.9797</td>
<td>9.0785e-02</td>
<td>1.7631</td>
</tr>
<tr>
<td>2</td>
<td>1.5327e-01</td>
<td>1.0046</td>
<td>2.2940e-02</td>
<td>1.9846</td>
</tr>
<tr>
<td>3</td>
<td>7.5521e-02</td>
<td>1.0211</td>
<td>4.5256e-03</td>
<td>2.3417</td>
</tr>
</tbody>
</table>

### Example 4.3 (Two-Asset European Option)

Let us now consider the application of our method to 2D FBS equations in the following three examples: Call-on-Min, Put-on-Max and Basket options. The system parameters for all these options are contained in Table 4.4. The 2D FBS equation: We consider the solution of (4.1a), with the market parameters in Table 4.4. There are several types of Rainbow Options.

For a Call-on-Min option, the terminal and boundary conditions are:

\[
U(x, y, T) = \min(e^x, e^y) - K,
\]

\[
U(x, y, t) = \min(e^x, e^y) - Ke^{r(T-t)}, \quad x = x_{\min}, x_{\max} \quad \text{or} \quad y = y_{\min}, y_{\max}.
\]

For a Put-on-Max option:

\[
U(x, y, T) = K - \max(e^x, e^y),
\]

\[
U(x, y, t) = Ke^{r(T-t)} - \max(e^x, e^y), \quad x = x_{\min}, x_{\max} \quad \text{or} \quad y = y_{\min}, y_{\max}.
\]
A Basket option is an exotic option whose underlying is a weighted sum or average of different assets. We use the average of the two assets here, so the terminal and boundary conditions are:

\[
U(x, y, T) = \left( \frac{e^x + e^y}{2} - K \right)_+, \\
U(x, y, t) = \left( \frac{e^x + e^y}{2} - Ke^{-r(T-t)} \right)_+, \quad x = x_{\min}, x_{\max} \text{ or } y = y_{\min}, y_{\max}.
\]

To solve this problem, we choose a mesh size \(M_x = M_y = 100\) and \(\Delta t = 1/100\). The numerical solutions for \(\alpha = \beta = 1.5\) are plotted in Figure 4.1, 4.2, 4.3 against \(t\) and the original independent variable \(S_x = e^x\) and \(S_y = e^y\).

To see the influence of \(\alpha\) and \(\beta\) on the option prices, we solve the problem for four different sets of values \(\alpha = \beta = 1.3, 1.5, 1.7, 1.9\) and plot the differences between the numerical solutions of the FBS equations and the standard BS equations (i.e., \(\alpha = 2\)) at \(t = 0\) for Call-on-Min (4.4), Put-on-Max (4.5) and Basket Options (4.6).

From these figures, we see that the call prices increase as \(\alpha\) decreases when \(S\) is greater than a critical value, which is the same as the results in 1D. Therefore, our numerical
Chapter 4. European Two-Asset Options

Figure 4.2: Computed Prices of a Put-on-Min Option

Figure 4.3: Computed Prices of a Basket Option
results are consistent with those from [9]. It is also seen that when $\alpha$ and $\beta$ approach 2, the numerical solutions to the FBS equations approach to those from the BS equations.

### 4.7 Conclusion

In this chapter, two numerical approaches (FDM and ADI) are proposed to solve the two-asset European option pricing problems. The existence and uniqueness of the solution for the continuous problem are proved through the variational formulation. The discretization
method is stable and convergent. We apply this method to solve two FPDE models and three different two-asset option pricing problems. These numerical results demonstrate the usefulness of our methods. Furthermore, An alternating direction implicit method is developed for the two-dimensional FPDEs to increase the computational efficiency.
Chapter 4. European Two-Asset Options

\[ \alpha, \beta = 1.3 \]

\[ \alpha, \beta = 1.5 \]

\[ \alpha, \beta = 1.7 \]

\[ \alpha, \beta = 1.9 \]

**Figure 4.6**: \( V_{bs} - V_{fbs} \) Basket Options
Chapter 5

American Two-Asset Options

The numerical solution of FBS equations arising in pricing an option written on single risky asset has been discussed in various existing works. To our best knowledge, there is no numerical method for the multi-dimensional FBS equations governing the valuation of options on multiple risky assets. In this Chapter, we propose a power penalty method for a 2DFBS equation arising in pricing American-style options on two underlying assets which follow two independent geometric Lévy processes. Unlike the European-style two-asset options in Chapter 4, the value of an American option is governed by LCP involving a 2DFBS operator $\mathcal{L}$ defined in Section 4.1 and a constraint on the option. The boundary conditions of the 2DLCP are determined by two LCPs as discussed in Chapter 3. We analyze the solvability of this continuous problem in Section 5.3.

To solve this 2DLCP, we first approximate it with a nonlinear 2D FPDE with a penalty term. We first prove that the solution to this nonlinear FPDE converges to that of the LCP in a Sobolev norm at an exponential rate which depends on the penalty parameters. And then we discretize the 2D nonlinear FPDE by a finite difference method based on a damped Newton’s method for the nonlinear term, Crank-Nicolson method for the time derivative and a 2nd-order discretization scheme developed in (2.19) for the $\alpha$ and $\beta$-th fractional partial derivatives.

In the end of the Chapter, the numerical experiment for pricing Basket options is performed to demonstrate the convergent rate and the effectiveness of our penalty method.
5.1 American Two-Asset Option Pricing Model

Similar to the single-asset American option, the two-asset American option prices follow a linear complementarity problem as below:

\[LU \geq 0, \quad (5.1a)\]
\[U \geq U^*, \quad (5.1b)\]
\[LU \cdot (U - U^*) = 0,\]

with boundary conditions

\[U(x_{\min}, y, t) = g_1(y, t), \quad U(x, y_{\min}, t) = g_2(x, t), \quad (5.1c)\]
\[U(x_{\max}, y, t) = 0, \quad U(x, y_{\max}, t) = 0,\]

and terminal conditions

\[U(x, y, T) = U^*(x, y), \quad (x, y) \in \Omega,\]

where \(\Omega = (x_{\min}, x_{\max}) \times (y_{\min}, y_{\max})\). The function \(g_1\) and \(g_2\) are determined via solving the associated one-dimensional American put option pricing. Here, we consider an American Basket option whose pay-off function is

\[U^*(x, y) = [K - w_1e^x - w_2e^y]_+ .\]

Now, we discuss about the boundary conditions for the above problem. The function \(g_1(y, t) = U(y, t)\) is to be determined on the boundary \(x = x_{\min}\), so the following equation must be satisfied.

\[
\begin{cases}
-L_t + a_2 U_y - b_2 \left[ -\infty D_y^2 U \right] + r U - \lambda [U^* - U]_+ \geq 0, \\
U \geq U^*(x_{\min}, y), \\
LU \cdot (U - U^*) = 0,
\end{cases}
\]

\(\text{(5.2)}\)
for \( x = x_{\text{min}} \), with the following boundary and terminal conditions:

\[
U(y_{\text{min}}, t) = K, \quad U(y_{\text{max}}, t) = 0, \\
U(y, T) = [(K - e^y)/w_2]_+.
\]

While the function \( g_2(x, t) \) is to be determined on the boundary \( y = y_{\text{min}} \), the following equation must be satisfied by \( U(x, t) = g_2(x, t) \).

\[
\begin{align*}
-\partial_t U + a_2 \partial_y U - b_1 \int_{-\infty}^y D_2^\alpha U + rU - \lambda [U^* - U]_+ & \geq 0, \\
U & \geq U^*(x, y_{\text{min}}), \\
\mathcal{L}U \cdot (U - U^*) & = 0,
\end{align*}
\]

with the following boundary and terminal conditions:

\[
U(x_{\text{min}}, t) = K, \quad U(x_{\text{max}}, t) = 0, \\
U(x, T) = [(K - e^x)/w_1]_+.
\]

The above cases are single-asset American option pricing problems, thus we can solve the equations (5.2) and (5.3) by the same method for 1D American put option in Chapter 3.

### 5.2 Variational Formulation and Unique Solvability

To analyze the solvability of the LCP (5.1) and convergent rate of the penalty method, we transform the (5.1) into a LCP with homogeneous Dirichlet boundary conditions. We use the transformation to let \( V = U_0 - U \), which is the same as in (4.2), where \( U_0(x, y) \in H^\gamma(\Omega) \) for all \( t \in (0, T] \), such that \( U_0 \) satisfied the boundary conditions given in (5.1c) and \( \nabla U_0 \) is continuous on \( \Omega \). We therefore get the following equations:

\[
\begin{align*}
\mathcal{L}V & \leq f, \quad (5.4a) \\
V & \leq V^*, \quad (5.4b) \\
(\mathcal{L}V - f) \cdot (V - V^*) & = 0,
\end{align*}
\]
Chapter 5. American Two-Asset Options

with the boundary and terminal conditions:

\[ V(x, y, t)|_{\partial \Omega} = 0, \]
\[ V(x, y, T) = V^*(x, y). \] (5.4c)

We define the following set first.

\[ \mathcal{K} = \{ v(t) : v(t) \in H^0_0(\Omega), v(t) \leq V^*(t) \text{ a.e. in } (0, T) \}. \]

It is easy to verify \( \mathcal{K} \) is a convex and closed subset of \( H^0_0(\Omega) \).

Using the notation defined above, we pose the following problem:

**Problem 5.1.** Find \( u \in \mathcal{K} \), such that, for all \( v \in \mathcal{K} \),

\[ \left\langle -\frac{\partial u}{\partial t}, v - u \right\rangle + A(u, v - u) \geq (f, v - u), \] (5.5)

almost everywhere (a.e) in \((0, T)\), satisfying terminal condition (5.4c), where \( A(\cdot, \cdot) \) is a bilinear form defined in (4.4).

Using a standard argument, it can be easily shown that Problem 5.1 is the variational form of (5.4a).

**Theorem 5.2.** There exists a unique solution to equation (5.4a) with boundary and payoff conditions (5.4c).

The coercivity and continuity of the bilinear form \( A(\cdot, \cdot) \) is proved in Chapter 4. While both of these guarantee that Problem 5.4a is uniquely solvable, thus the proof is omitted here.

### 5.3 Penalized Equation and Convergence

The penalized equation to solve American option pricing problem is given below:

\[ \mathcal{L}u_\lambda + \lambda [u_\lambda - u^*]_+^{1/k} = f, \quad (x, y, t) \in \Omega \times (0, T) \] (5.6a)
satisfying the following boundary and terminal conditions:

\[ u_\lambda(x,y,t)|_{\partial\Omega} = 0, \quad \text{(5.6b)} \]
\[ u_\lambda(x,y,T) = u^*(x,y), \quad \text{(5.6c)} \]

where \( \lambda > 1, \ k > 0 \) are penalty parameters. The payoff function for a Basket option is \( u^*(x,y) = [K - w_1 e^x - w_2 e^y]_+ \).

The above penalty equation transforms a linear complementarity problem into a nonlinear FPDE, and the term \( \lambda [u_\lambda(x,y,t) - u^*(x,y,t)]_+^{1/k} \) in (5.6a) penalizes the part of \( u_\lambda \), which violates the constraint (5.4b) when \( \lambda \) and \( k \) is sufficiently large. The equivalent variational form of (5.6a) is as follows.

**Problem 5.3.** Find \( u(t) \in H^{7/2}_0(\Omega) \), such that, for all \( v \in H^{7/2}_0(\Omega) \),

\[
\left\langle -\frac{\partial u(t)}{\partial t}, v \right\rangle + A(u(t), v) + \left( \lambda [u(x,t) - u^*(x)]_+^{1/k}, v \right) = (f(t), v)
\]

almost everywhere (a.e) in \((0,T)\) satisfying terminal condition (5.6c) a.e. in \((x_{min}, x_{max}) \times (y_{min}, y_{max})\), where \( A(\cdot, \cdot) \) is a bilinear form defined in (4.4).

**Theorem 5.4.** Problem 5.3 has a unique solution.

**Proof.** It suffices to show that the nonlinear operator on the LHS of (5.6a) is strongly monotone and continuous.

Moreover, from (4.6) we see that \( A(u,v) \) is Lipschitz continuous in both \( u \) and \( v \). Also, it is obvious that \( [u - u^*]_+^{1/k} \) is continuous in both \( u \) and \( v \). Therefore, Problem 5.3 is uniquely solvable by the standard result in [25, Page 37].

5.3.1 Convergence Analysis

We will show that the solution to Problem 5.3 converges to that of (5.4a) as the penalty parameter \( \lambda \to \infty \) with exponential order convergence rate under some proper norms.

Before further discussion, it is necessary to introduce the usual Hilbert space in space and time and its \( L^p \)-norm defined respectively by

\[ L^p(0,T; H^q_0(\Omega)) := \{ v(\cdot, \cdot, t) : v(\cdot, \cdot, t) \in H^q_0(\Omega) \ \text{a.e. in} \ (0,T); \|v(\cdot, \cdot, t)\|_\gamma \in L^p((0,T)) \} \]
The norm of $L^p(0, T; H(\Omega))$ is denoted by

$$\|v(\cdot, \cdot, t)\|_{L^p(0, T; H(\Omega))} = \left( \int_0^T \|v(\cdot, \cdot, t)\|_{H(\Omega)}^p \right)^{1/p}.$$ 

We first give the following lemma.

**Lemma 5.5.** Let $u_\lambda$ be the solution to Problem 5.3 and assume that $u_\lambda \in L^2(\Omega \times (0, T))$. Then there exists a positive constant $C$, independent of $u_\lambda$ and $\lambda$, such that

$$\|[u_\lambda - u^*]_+\|_{L^p(\Omega \times (0, T))} \leq \frac{C}{\lambda^{k/2}}, \quad (5.7)$$

$$\|[u_\lambda - u^*]_+\|_{L^\infty(0, T; L^2(\Omega))} + \|[u_\lambda - u^*]_+\|_{L^2(0, T; H^{\gamma/2}_0(\Omega))} \leq \frac{C}{\lambda^{k/2}}. \quad (5.8)$$

**Proof.** Let $C$ is a generic positive constant, independent of $u_\lambda$ and $\lambda$. We let $\phi(x, y, t) = [u_\lambda(x, y, t) - u^*(x, y)]_+$ for simplicity. It is easy to see that $\phi(\cdot, \cdot) \in H^{\gamma/2}_0(\Omega)$ for $t \in (0, T)$ a.e. Thus, setting $v = \phi$ in (5.3), we have

$$\left\langle -\frac{\partial u_\lambda}{\partial t}, \phi \right\rangle + A(u_\lambda, \phi) + \lambda \left( \phi^{1/k}, \phi \right) = (f, \phi) \quad \text{a.e. in } (0, T).$$

Taking $-\left\langle \frac{\partial u^*}{\partial t}, \phi \right\rangle + A(u^*, \phi)$ away from both sides of the above equality gives

$$\left\langle -\frac{\partial(u_\lambda - u^*)}{\partial t}, \phi \right\rangle + A(u_\lambda - u^*, \phi) + \lambda \left( \phi^{1/k}, \phi \right) = (f, \phi) + \left\langle \frac{\partial u^*}{\partial t}, \phi \right\rangle - A(u^*, \phi),$$

or

$$\left\langle -\frac{\partial \phi}{\partial t}, \phi \right\rangle + A(\phi, \phi) + \lambda \left( \phi^{1/k}, \phi \right) = (f, \phi) - A(u^*, \phi), \quad (5.9)$$

since $\phi = 0$ when $u_\lambda - u^* < 0$ and $\frac{\partial u^*}{\partial t} = 0$. Note $\phi(x, y, T) = [u_\lambda(x, y, T) - u^*(x, y)]_+ = 0$ by (5.6b).

Noticing that

$$\int_t^T \left\langle -\frac{\partial \phi(t)}{\partial t}, \phi(\tau) \right\rangle d\tau = (\phi(t), \phi(t)) - \int_t^T \left\langle -\frac{\partial \phi(t)}{\partial t}, \phi(\tau) \right\rangle d\tau,$$

since $\phi(T) = 0$. From this, we get

$$\int_t^T \left\langle -\frac{\partial \phi(t)}{\partial t}, \phi(\tau) \right\rangle d\tau = \frac{1}{2} (\phi(t), \phi(t)) \geq 0. \quad (5.10)$$
Integrating (5.9) from \( t \) to \( T \) and using (5.10), (4.5) and Hölder Inequality, we get
\[
\frac{1}{2} (\phi(t), \phi(t)) + C \int_t^T \| \phi(\tau) \|_{\gamma/2}^2 \, d\tau + \lambda \int_t^T \| \phi(\tau) \|_{L^p(I)}^p \, d\tau \\
\leq \int_t^T (f(\tau), \phi(\tau)) \, d\tau - \int_t^T A(u^*, \phi(\tau)) \, d\tau \\
\leq C \left( \int_t^T \| \phi(\tau) \|_{L^p(I)}^p \, d\tau \right)^{1/p} - \int_t^T A(u^*, \phi(\tau)) \, d\tau. \tag{5.11}
\]

Let \( \zeta = \gamma - 1 > 0 \), from the definition of \( A(\cdot, \cdot) \) in (4.4), we see that the integrand of the last term in (5.11) is
\[
- A(u^*, \phi) = (au^* + B\nabla^{(\zeta)} u^*, \nabla \phi) + r(u^*, \phi).
\]

By Green’s Theorem, we have
\[
- \int_t^T (au^*, \nabla \phi) \, d\tau = \int_t^T \int_{\Omega} \nabla \cdot (au^*) \phi \, d\Omega \, d\tau - \int_t^T \int_{\partial \Omega} (u^* \cdot \nabla \phi) \, d\Gamma \, d\tau \\
\leq C \int_t^T \int_{\Omega} d\Omega \, d\tau \leq C \left( \int_t^T \| \phi(\tau) \|_{L^p(\Omega)}^p \right)^{1/p},
\]
because \( \nabla \cdot au^* \) is bounded above on \( \Omega \).

Let \( \Omega_1 = \{ 0 < x < K, 0 < y < K, K - \max[x, y] > 0 \} \) and \( \Omega_2 = \Omega \setminus \overline{\Omega}_1 \). Let \( \Gamma_0 \) denote the interface of \( \Omega_1 \) and \( \Omega_2 \). So, \( \Gamma_0 \) has two opposite orientations: \( \Gamma_0^+ \) which is oriented in the same direction as \( \partial \Omega_1 \), and \( \Gamma_0^- \) which is oriented in the same direction as \( \partial \Omega_2 \). Since \( \phi = 0 \) is on \( \Gamma \), for \( \phi \in H^1_0(\Omega) \), we have
\[
-(B\nabla^{(\zeta)} u^*, \nabla \phi) \\
= - \int_{\Omega} (B\nabla^{(\zeta)} u^*)^T \nabla \phi \, d\Omega \\
= - \int_{\Omega_1} (B\nabla^{(\zeta)} u^*)^T \nabla \phi \, d\Omega - \int_{\Omega_2} (B\nabla^{(\zeta)} u^*)^T \nabla \phi \, d\Omega \\
= \int_{\Omega_1} \nabla \cdot (B\nabla^{(\zeta)} u^*) \phi \, d\Omega - \int_{\Gamma_0^+} B\nabla^{(\zeta)} u^* \cdot \nabla \phi \, ds \\
+ \int_{\Omega_2} \nabla \cdot (B\nabla^{(\zeta)} u^*) \phi \, d\Omega - \int_{\Gamma_0^-} B\nabla^{(\zeta)} u^* \cdot \nabla \phi \, ds \\
= - \int_{\Gamma_0^+} (B\nabla^{(\zeta)} u^*_+ - B\nabla^{(\zeta)} u^*_-) \cdot \nabla \phi \, ds + \int_{\Omega} \nabla \cdot (B\nabla^{(\zeta)} u^*) \phi \, d\Omega, \tag{5.12}
\]
where \( n \) is the unit outward normal direction of the boundary and \( \nabla u^* - \nabla u^* + \) denote the value of \( \nabla u^* \) on the left and right sides of \( \Gamma_0^+ \) respectively. Since \( u^* = V_0 - V^* \), thus

\[

\nabla u^* = \nabla V_0 - \nabla V^* ,
\]

which implies that

\[

\nabla u^*_+ - \nabla u^*_+ = \nabla V^*_+ - \nabla V^*_+ = (-\omega_1, -\omega_2).
\]

The unit outward-normal vector to \( \Gamma_0^+ \) is

\[
n = \frac{\nabla (K - \max\{x, y\})}{\|\nabla (K - \max\{x, y\})\|} = (-\omega_1, -\omega_2)^T/(\omega_1^2 + \omega_2^2).
\]

So, (5.12) can be written as follows:

\[

- (B\nabla (\zeta u^*), \nabla \phi) - \int_{\Gamma_0^+} (\omega_1, \omega_2)B^T(\omega_1, \omega_2)^T \frac{\phi ds}{\omega_1^2 + \omega_2^2} + \int_{\Omega} \nabla \cdot (B\nabla (\zeta u^*)) \phi d\Omega
\]

\[
\leq C \int_{\Omega} \phi(\tau) d\Omega,
\]

since \( B \) is positive definite, \( \phi \) is non-negative and \( \nabla \cdot (B\nabla (\zeta u^*)) \) is bounded above on \( \Omega \).

Therefore, replacing the last term in (5.11) by the above upper bound gives

\[
\frac{1}{2} (\phi(t), \phi(t)) + C \int_{t}^{T} \|\phi(\tau)\|^2_{L^2} d\tau + \lambda \int_{t}^{T} \|\phi(\tau)\|^p_{L^p(I)} d\tau \leq C \left( \int_{t}^{T} \|\phi(\tau)\|^p_{L^p(I)} d\tau \right)^{1/p},
\]

for all \( t \in (0, T) \). This implies that

\[
\lambda \int_{t}^{T} \|\phi(\tau)\|^p_{L^p(I)} d\tau \leq C \left( \int_{t}^{T} \|\phi(\tau)\|^p_{L^p(I)} d\tau \right)^{1/p},
\]

or

\[
\left( \int_{t}^{T} \|\phi(\tau)\|^p_{L^p(I)} d\tau \right)^{1-1/p} \leq C\lambda^{-1}.
\]

Since \( 1 - 1/p = 1/(kp) \), it follows from the above estimate that

\[
\left( \int_{t}^{T} \|\phi(\tau)\|^p_{L^p(I)} d\tau \right)^{1/p} \leq C\lambda^{-k}.
\]
Chapter 5. American Two-Asset Options

This is (5.7). Combining (5.13) yields

\[
\frac{1}{2}(\phi(t), \phi(t)) + \int_t^T \|\phi(\tau)\|_{\alpha/2}^2 d\tau \leq C \frac{\lambda}{k}.
\]

Finally, the above inequality implies (5.8).

Using Lemma 5.5, we establish the main convergence result in the following theorem.

**Theorem 5.6.** Let \( u \) and \( u_\lambda \) be the solutions to Problems 5.5 and 5.3, respectively. If \( \frac{\partial u}{\partial t} \in L^2(\Omega \times (0,T)) \), then there exists a constant \( C > 0 \), independent of \( \lambda \) and \( k \), such that

\[
\|u_\lambda - u\|_{L^\infty(0,T;L^2(\Omega))} + \|u_\lambda - u\|_{L^2(0,T;H^{\gamma/2}(\Omega))} \leq C \frac{\lambda}{k^{1/2}},
\]

where \( \lambda \) and \( k \) are the parameters used in (5.6a).

**Proof.** Since the proof is similar to Theorem 3.7, thus it is omitted here. \( \square \)

### 5.4 Discretization

In this section, a numerical scheme is proposed to solve this 2D linear complementarity problem (5.1). We also use Crank-Nicolson time stepping method and the same finite difference method to construct the discretization scheme for the penalized equation of (5.1a) as follows.

\[
\begin{align*}
\frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t} &+ \frac{1}{2} \left( a_1 \delta_x U_{i,j}^{n+1} - b_1 \delta_x U_{i,j}^{n+1} + a_2 \delta_y U_{i,j}^{n+1} - b_2 \delta_y U_{i,j}^{n+1} + r U_{i,j}^{n+1} - d_{ij}^{n+1} \right) \\
&+ \frac{1}{2} \left( a_1 \delta_x U_{i,j}^n - b_1 \delta_x U_{i,j}^n + a_2 \delta_y U_{i,j}^n - b_2 \delta_y U_{i,j}^n + r U_{i,j}^n - d_{ij}^n \right) \\
&= \frac{1}{2} \left( f_{ij}^n + f_{ij}^{n+1} \right),
\end{align*}
\]

for \( i = 1, 2, ..., M_x - 1 \), \( j = 1, 2, ..., M_y - 1 \) and \( n = 1, 2, ..., N \) satisfying

\[
\begin{align*}
U_{0,j}^n &= g_1(y_j, t_n), & U_{i,0}^n &= g_2(x_i, t_n), \\
U_{i,j}^0 &= U^*(x_i, y_j),
\end{align*}
\]

89
Chapter 5. American Two-Asset Options

where \( d_{ij} = d(U_{ij}^n) = [U^* - U_{ij}^n]^{1/k} \) is the penalty term. From (5.15a), we get the full discrete system as follows:

\[
\left( I + \frac{1}{2}M \right) V^{n+1} + \frac{1}{2}D(V^{n+1}) = \left( I - \frac{1}{2}M \right) V^n - \frac{1}{2}D(V^n) + \bar{F}^n.
\]

where

\[
D(V^n) = \left( d(U_{1,1}^n), d(U_{2,1}^n), ..., d(U_{M_x-1,1}^n), ..., d(U_{M_x-1,M_y-1}^n) \right)^T,
\]

for \( n = 0, 1, ..., N - 1 \), where \( M \) is the matrix defined in (4.8). In (5.15b), the boundary conditions \( g_{1j}^n \) and \( g_{2i}^n \) are the numerical solutions from two 1D systems.

Using damped Newton’s iterative method to solve the nonlinear system (5.15), we have

\[
\left( I + \frac{1}{2}M + \frac{1}{2}J_D \left( w^{l-1} \right) \right) \delta w^{l-1} = \left( I - \frac{1}{2}M \right) V^n - \frac{1}{2}D(V^n) + \bar{F}^n - \left( I + \frac{1}{2}M \right) w^{l-1} - \frac{1}{2}D \left( w^{l-1} \right)
\]

\[
w^{l} = w^{l-1} + \kappa \delta w^{l-1}
\]

for \( l = 1, 2, ... \) with \( w^{0} = V^n \) being the initial guess, where \( J_D(w) \) denotes the Jacobian matrix of the column vectors \( D(w) \) and \( \kappa \in (0, 1] \) denotes the damping parameter. Then we choose

\[
V^{n+1} = \lim_{l \to \infty} w^{l},
\]

for all \( n = 0, 1, 2, ..., N - 1 \).

5.5 Numerical Results

In this section, we solve an American basket option pricing problem to demonstrate the accuracy and usefulness of our numerical method.

Example 5.1. We demonstrate the efficiency and usefulness of the penalty method by solving the weighted American put option pricing problem with different values of \( \lambda \) and \( k \). We consider the solution of (5.1), with market parameters given in Table 5.1. The weights of assets \( x \) and \( y \) are \( w_1 = w_2 = 0.5 \).
Chapter 5. American Two-Asset Options

### Table 5.1: Market Parameters for a Two-Asset American Option

<table>
<thead>
<tr>
<th>Parameters Values</th>
<th>$\alpha$, $\beta$</th>
<th>1.5</th>
<th>$r$</th>
<th>0.05</th>
<th>$\sigma$</th>
<th>0.25</th>
<th>$K$</th>
<th>30</th>
<th>$a_1$, $a_2$</th>
<th>0.384</th>
<th>$b_1$, $b_2$</th>
<th>0.884</th>
<th>$x_{\text{min}}$, $y_{\text{min}}$</th>
<th>ln 0.1</th>
<th>$x_{\text{max}}$, $y_{\text{max}}$</th>
<th>ln 100</th>
</tr>
</thead>
</table>

Table 5.2: Convergence behaviour with increasing $\lambda$.

To investigate the convergent rate of the method in both $\lambda$ and $k$, we choose a fixed uniform mesh for the solution domain $(\ln(0.1), \ln(100)) \times (0, 1)$ in $(x, t)$ with $M_x = M_y = 50$ and $N = 50$.

Since there is no exact solution to this problem, we use the numerical solution with $\lambda = 10^6$ and $k = 1$ as the reference solution denoted as $V_R$. Using this reference solution, we solve (5.6a) on the aforementioned uniform mesh for a sequence of values of $\lambda$ when $k$ is fixed and compute the corresponding approximations of

$$\|V_R - V_\lambda\|_{L^\infty(0,T;L^2(\Omega))} + \|V_R - V_\lambda\|_{L^2(0,T;H^{\gamma/2}(\Omega))}$$

on these meshes.

To investigate computational rate of convergence of the method in $k$ for a fixed $\lambda$, from (5.14) we see the ratio of the errors in the solutions from $k$ and $k+1$ equals $O(\lambda^{k+1}/\lambda^k) = O(\lambda)$. The computed results for different values of $k$ and $\lambda$ are listed in Table 5.3, from which we see that the ratios of the errors for any two consecutive values of $k$ are almost constants.

We have also repeated the above numerical experiments for $\alpha = \beta = 1.3, 1.7$ and found that the computed convergent rates are the same as the corresponding ones for $\alpha = \beta = 1.5$. Thus, the convergent rates of the penalty method does not depend on the fractional order $\alpha$ or $\beta$. 

91
**Figure 5.1:** Computed Prices of an American Basket Option when $\alpha = 1.5$

<table>
<thead>
<tr>
<th>$\lambda = 10 \times 2^n$</th>
<th>$k = 1$</th>
<th>$k = 2$</th>
<th>$k = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 20$</td>
<td>Error</td>
<td>1.3229</td>
<td>1.6211</td>
</tr>
<tr>
<td></td>
<td>log 2Ratio</td>
<td>0.8160</td>
<td>0.8673</td>
</tr>
<tr>
<td>$\lambda = 40$</td>
<td>Error</td>
<td>0.6857</td>
<td>0.4752</td>
</tr>
<tr>
<td></td>
<td>log 2Ratio</td>
<td>1.4430</td>
<td>1.4216</td>
</tr>
<tr>
<td>$\lambda = 80$</td>
<td>Error</td>
<td>0.3493</td>
<td>0.1246</td>
</tr>
<tr>
<td></td>
<td>log 2Ratio</td>
<td>2.8025</td>
<td>2.7904</td>
</tr>
<tr>
<td>$\lambda = 160$</td>
<td>Error</td>
<td>0.1763</td>
<td>0.0315</td>
</tr>
<tr>
<td></td>
<td>log 2Ratio</td>
<td>5.5900</td>
<td>5.6529</td>
</tr>
</tbody>
</table>

**Table 5.3:** Convergence behaviour with increasing $k$. 
To see the influence of $\alpha$ and $\beta$ on the option prices, we solve the problem for $\alpha = \beta = 1.3, 1.5, 1.7$, and plot the section when $S_x = S_y$ in Figure 5.2 the differences between the numerical solutions of the FBS equation and the standard BS equation (i.e., $\alpha = 2$) $V_{FBS} - V_{BS}$ at $t = 0$. From Figure 5.2, we see that the American put option from the fBS model is more valuable than that from the standard BS model. Also, the value of the option increases as $\alpha$ and $\beta$ decreases.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Basket Option Comparison for Different $\alpha$}
\end{figure}

5.6 Conclusion

In this Chapter, we proposed a penalized FPDE involving a two-dimensional fractional Black-Scholes operator to approximate a linear complementarity problem on two-asset American option pricing. We proposed a power penalty method for the numerical solution of the 2D FBS equation and show that the solution of the penalized equation converges to the LCP at an exponential order. Combining the finite difference method developed in Chapter 4 and a damped Newton’s method, we solved an American basket option pricing
problem numerically. The numerical results illustrate the usefulness and accuracy of the numerical scheme which is consistent with the theoretical results.
Chapter 6

Conclusions and Future Research

6.1 Conclusions

In this thesis, we have developed numerical methods for pricing European, American, and two-asset options whose underlying assets follow Lévy processes.

We proposed a second order finite difference methods for solving the one-dimensional FBS equation governing European single-asset option pricing. We show the unique solvability of both the continuous and discretized FBS equations and establish the convergence of the numerical solution to the viscosity solution of the continuous FBS equation by proving the consistency, stability and monotonicity of the FDM.

We proposed a power penalty method for a fractional-order differential LCP governing American option pricing. We proved that the penalty methods have an exponential convergence rate depending on the choices of the penalty parameters.

The discretization and penalty methods and their analysis were further developed and applied to the two-dimensional FBS equation and LCP arising in pricing two-asset European and American options. To reduce the computational cost, an ADI method is proposed for solving the discretized two-dimensional FBS equation.

We performed numerical tests on all the methods proposed in this thesis to demonstrate the theoretical results. The numerical results showed that the discretization methods had 2nd-order accuracy, while the GL and L2 methods only had 1st-order accuracy.
The other numerical results showed exponential convergence rate of our penalty methods and high efficiency of the ADI method. Our methods also presented accurate, stable and financially meaningful results.

6.2 Future Research

There are several further research directions in which these numerical methods proposed in this thesis can be extended.

Firstly, the underlying assets can be also assumed to follow another two famous Lévy processes: the KoBoL process and CGMY process, and the corresponding FBS equations derived for these two models contain two-sided spatial fractional derivatives $-\infty D_{x}^{\alpha}$ and $\infty D_{x}^{\alpha}$. Thus, implementing our finite difference methods to these FBS equations can be also very meaningful.

Secondly, the power penalty methods can also be applied to the fractional-order LCP involving the two-sided fractional derivative.

Thirdly, because computational efficiency of numerical experiments in Chapter 5 is so low, that this method is hard to implement for higher dimensional FPDEs directly. Seeking for a proper ADI method for higher (> 3) dimensional FPDE to increase the computational efficiency is necessary.

At last, other numerical methods e.g. Monte Carlo method and Fast Fourier transform are worth investigating in solving fractional PDEs.
Bibliography


