\[ I_D(\tau) \propto |E_s(t) + E_R(t)|^2. \] (2-7)

Substituting Eq. (2-6) into Eq. (2-7), the irradiance at the detector can then be expressed as a function of \( \tau \) by interference of a time-delayed version of the source light with itself:

\[ I_D(\tau) \propto \sqrt{R_s E(t) + R_R E(t + \tau)} = R_s |E(t)|^2 + R_R |E(t + \tau)|^2 + 2\sqrt{R_s R_R} \text{Re}\{E^*(t)E(t + \tau)\}, \] (2-8)

where \( \text{Re}\{a\} \) is the real part of \( a \).

The third term of Eq. (2-8) contains the self-coherence function of the source (see Eq. (2-4)). So we can rewrite Eq. (2-8) as:

\[ I_D(\tau) = I_0 (R_s + R_R) + 2\sqrt{R_s R_R} \text{Re}\{\Gamma(\tau)\} = I_0 (R_s + R_R) + 2\sqrt{R_s R_R} \text{Re}\{\gamma(\tau)\} \] (2-9)

where \( \gamma(\tau) = \Gamma(\tau)/\Gamma(0) \) is the complex degree of coherence, or simply, the coherence function, of the light, and \( \Gamma(0) = I_0 \) [34]. \( \gamma(\tau) \) has the properties [34]

\[ \gamma(0) = 1 \quad \text{and} \quad |\gamma(\tau)| \leq 1. \] (2-10)

Considering a normalized Gaussian power spectral density, given by:

\[ g(\nu) = \frac{2}{\Delta \nu} \sqrt{\frac{\ln 2}{\pi}} e^{-\left(2\sqrt{\ln 2} \frac{\nu - \nu_0}{\Delta \nu}\right)^2}, \] (2-11)

where \( \nu_0 \) is the centre optical frequency, the corresponding coherence function is

\[ \gamma(\tau) = e^{-\left(\frac{\Delta \nu \tau}{\sqrt{\ln 2}}\right)^2} e^{-i2\pi\nu_0 \tau}. \] (2-12)

As observed in Figure 2-13, the properties in Eq. (2-10) are verified.

For a monochromatic wave, we can define the wave velocity as its phase velocity \( v_p = \omega / k \). For a polychromatic wave, we can define a phase velocity for each spectral component and a group velocity for the wave, \( v_g = \partial \omega / \partial k \). In the case of air, both velocities are equal and closely approximate \( c \).

The delay \( \tau = \Delta L/c \) represents the time light takes to travel the optical path length difference \( \Delta L \), where \( \Delta L = 2|n_R z_R - n_z z_s| \), \( z_R \) and \( z_s \) are the distances from the beam splitter to the reference reflector and to the sample reflector, respectively, \( c \) is the speed of light, and \( n_R \) and \( n_s \) are the reference reflector and sample medium group refractive indexes, respectively. The group refractive index is the ratio of the velocity of light in
vacuum to the group velocity in the medium. The corresponding phase delay is 
\[ \tau_p = \frac{\Delta L}{v_p} \]
and group delay is 
\[ \tau_g = \frac{\Delta L}{v_g}. \]

Detecting interference fringes on the femtosecond time scale is impracticable because electrons in detectors cannot respond on such time scales, necessary to directly detect the electric field, but that is where the beating of two electric fields at slightly different frequencies helps in shifting the detection centre frequency and bandwidth to lower values, where the detection electronics can respond. The reason the field returning from the reference arm is frequency shifted is the Doppler shift introduced in the reference arm field by the moving mirror.

Assuming the reference arm mirror is scanned over an A-scan period \( \Delta t \), with constant velocity \( v_m = \frac{\Delta L}{\Delta t} \), the Doppler shift and the frequency beating result in a detectable fringe frequency downshifted by the ratio of the mirror velocity to the phase velocity from twice the centre optical frequency:

\[ f_0 = 2v_0 \left( \frac{v_m}{v_p} \right) \equiv 2v_0 \left( \frac{v_m}{c} \right). \] (2-13)

Figure 2-15. Interferometry with a monochromatic wave. The Doppler shift of the reference arm wave, generated by the translating mirror, results in wave beating with a carrier at the mean frequency between the original and the Doppler-shifted one, and a sinusoidal envelope with frequency equal to half of the Doppler frequency. The photodetector output follows the sinusoidal envelope due to its slow (compared to optical frequencies) response time. Mirror speed exaggerated: \( v = 5,000 \text{ km/s} \); carrier frequency: \( f = 354 \text{ THz} \); sinusoidal envelope (fringe) frequency = 12 THz.